Chapter 1: Prerequisites for Calculus

1.1: Lines

1) Slope $= m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$

2) Vertical Lines - $\Delta x = 0$, No slope

3.) Horizontal Lines - $\Delta y = 0$, $m = 0$

4.) Parallel Lines: $m_1 = m_2$ (slopes are equal)

5.) Perpendicular Lines: $m_2 = \frac{-1}{m_1}$ or $m_1 \cdot m_2 = -1$

Equations of Lines

1.) Vertical line through (a, b): $x = a$

2.) Horizontal line through (a, b): $y = b$

Find the vertical and horizontal lines through $(5, 7)$

Vertical line: $\boxed{x = 5}$ Horizontal line: $\boxed{y = 7}$

3.) Point-Slope Equation: $\boxed{y - y_1 = m(x - x_1)}$

or

$y = m(x - x_1) + y_1$

Find the equation of the line through the point $(4, 5)$ with a slope of $-3/2$.

$$y - 5 = -\tfrac{3}{2}(x - 4)$$
$$y = -\tfrac{3}{2}x + 6 + 5$$
$$\boxed{y = -\tfrac{3}{2}x + 11}$$

4.) Slope-Intercept Equation: $y = mx + b$

5.) General Linear Equation: $Ax + By = C$

Standard form: $Ax + By + C = 0$

6.) Point-to-Point Distance: $d = \sqrt{\Delta x^2 + \Delta y^2}$

7.) Point-to-Line distance: $d = \frac{|Ax+By+C|}{\sqrt{A^2+B^2}}$

In class examples:

1.) Find an equation that relates Fahrenheit and Celsius temperatures.

2.) Do a programming example: We want to use a "For Loop" to print some information from an array. We need to find a formula that will allow us to print on line 5 when our variable is 1, line 8 when $x=2$. See the table below:

Variable x	line to print on
1	5
2	8
3	11
4	14
5	17

This is Known as Regression Analysis

Points (x,y): $(1,5)$ + $(2,8)$

$$m = \frac{8-5}{2-1} = 3$$

$$y-5 = 3(x-1)$$

$$y-5 = 3x-3$$

$$\boxed{y = 3x+2}$$

The Program:

```
For X = 1 to 5
    Locate 3*X+2
    Print N$(Q, X)
Next X
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Assignment: Pages 9-11 (13, 15, 17, 19, 21, 23, 27, 29, 32, 34, 37, 41, 54a)

1.2 Functions and Graphs

Function: A function from a set D to a set R is a rule that assigns a unique element in R to each element in D.

Domain: the independent values - the "x" values.
Range: the dependent values - the "y" values

4 Types of Domain + Range

1.) Open Interval: $1 < x < 2$ $\quad (1,2)$

2.) Closed Interval: $1 \leq x \leq 2$ $\quad [1,2]$

3.) Half-open: $1 < x \leq 2$ $\quad (1,2]$

4.) Infinite Interval: $-\infty < x \leq 2$ $\quad (-\infty,2]$

This is called Interval Notation

Even + Odd functions

Even function: $f(-x) = f(x)$
Ex: $y = x^2$ $\rightarrow$ Symmetry about the y-axis

Odd function: $f(-x) = -f(x)$
Ex: $y = x^3$ $\rightarrow$ Symmetry about the origin

Absolute Values

$$|x| = \sqrt{x^2}$$

Also defined by the piecewise (or multi-part) function:

$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

Solve: $|x| + |x-1| = 3$

$$\boxed{x = 2 \text{ or } x = -1}$$

Finding Domain + Range

Examples for Class:

1.) $y = x + 3$

2.) $y = \sqrt{x+3}$

3.) $y = \dfrac{1}{x+3}$

4.) $y = \dfrac{1}{\sqrt{x+3}}$

Assignment: Page 19 (1-4, 5, 7, 9, 31, 33)

Look at page 20 (56)

1.3 Exponential Functions

Definition of Exponential Function

Let a be a positive real number other than 1.

$$f(x) = a^x$$

is the exponential function with base a.

Rules for Exponents ($a + b > 0$)

1.) $a^x \cdot a^y = a^{x+y}$

2.) $\dfrac{a^x}{a^y} = a^{x-y}$

3.) $(a^x)^y = (a^y)^x = a^{xy}$

4.) $a^x \cdot b^x = (a \cdot b)^x$

5.) $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

Exponential Growth: $y = P \cdot a^x$, where

P = initial investment/value
a = 1 plus interest/growth rate
x = number of years

Example of Population Growth

Year	Population in millions	Ratio
2002	287.9	290.4/287.9 ≐ 1.0087
2003	290.4	293.2/290.4 ≐ 1.0096
2004	293.2	295.9/293.2 ≐ 1.0092
2005	295.9	298.8/295.9 ≐ 1.0098
2006	298.8	301.6/298.8 ≐ 1.0094
2007	301.6	

From the above ratio:
$(1.0087 + 1.0096 + 1.0092 + 1.0098 + 1.0094)/5$
we conclude that the average would be 1.00934.

Can we provide a "model" for the population in a given year?

What would the population be in 2012?

$$y = 287.9 \cdot (1.00934)^{10}$$

10 years 2012 (2002+10)

population in 2002

1 plus growth rate of .00934 (or .934%)

$$\boxed{y \doteq 315.95 \text{ million people}}$$

Predicting US Population
Using the data in the table below, come up with a formula to predict the population of the US in 2000. Compare the results to the actual population of 281.4 million

Year	Population (in millions)	Ratio
1880	50.2	63/50.2 = 1.25498
1890	63.0	76.2/63 = 1.20952
1900	76.2	92.2/76.2 = 1.20997
1910	92.2	106.0/92.2 = 1.14967
1920	106.0	123.2/106 = 1.16226
1930	123.2	132.1/123.2 = 1.07224
1940	132.1	151.3/132.1 = 1.14534
1950	151.3	179.3/151.3 = 1.18506
1960	179.3	203.3/179.3 = 1.13385
1970	203.3	226.5/203.3 = 1.11412
1980	226.5	248.7/226.5 = 1.09801
1990	248.7	

Average = 1.15773

$$y = Pa^x$$

$$y = (50.2)(1.15773)^{12}$$

(Look at their example # 5 on page 25)

$$\boxed{y = 291.06 \text{ million}}$$

over estimate of $\approx$ 9.7 million, or 3.4%

The Number "e"

$e = 2.718281828$

e is defined as the function

$$f(x) = \left(1 + \frac{1}{x}\right)^x \text{ as } x \text{ approaches } \infty$$

e is used in a number of way, but is often used to calculate continuously compounded interest.

$$\boxed{y = Pe^{rt}}$$

where P is the principle, r is the interestrate, and t is time in years.

Assign: Quick Review P26 (9, 10)

Exercises P26 (1, 3, 5, 7, 19, 20)

1.4 Parametric Equations

A <u>relation</u> is a set of ordered pairs (x, y).

The <u>graph of a relation</u> is a set of points in a plane that correspond to the ordered pair.

If x & y are functions of a third variable (t - called a parameter), then we have a parametric equation.

Find the Cartesian Equation for the Parametric Equation:

$$y = t \quad , \quad x = \sqrt{t}$$
$$t = x^2$$
$$\therefore \; y = x^2$$

Circle

$$(x-h)^2 + (y-k)^2 = r^2$$

r = radius

(h, k) = center

$$\boxed{x^2 + y^2 - 2x - 4y + 1 = 0}$$

Use completing the square to show this is the circle:

$$(x-1)^2 + (y-2)^2 = 4$$

Center at $(1,2)$

radius $= 2$

Ellipse

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

(h, k) = center

a & b are the radius of the major/minor axes on the x & y axes.

Find the Cartesian Equation for the following Parametric Equations and then graph:

$x = 3\cos(t)$, $y = 3\sin(t)$

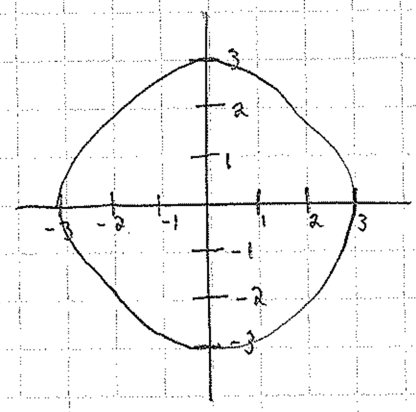
$x^2 = 9\cos^2(t)$, $y^2 = 9\sin^2(t)$

$x^2 + y^2 = 9\cos^2(t) + 9\sin^2(t)$

$x^2 + y^2 = 9(\cos^2(t) + \sin^2(t))$

$\boxed{x^2 + y^2 = 9}$

circle center (0,0)
radius = 3



Assign: Page 33 (5, 6, 9, 12, 14, 15)

Find the Cartesian Equation for each.

Ellipses

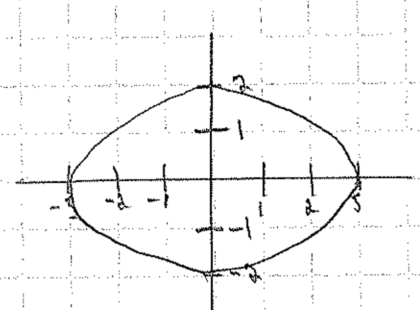
$x = 3\cos(t)$, $y = 2\sin(t)$

$\frac{x}{3} = \cos(t)$, $\frac{y}{2} = \sin(t)$

$\frac{x^2}{9} = \cos^2(t)$ $\frac{y^2}{4} = \sin^2(t)$

$\frac{x^2}{9} + \frac{y^2}{4} = \cos^2(t) + \sin^2(t)$

$\boxed{\frac{x^2}{9} + \frac{y^2}{4} = 1}$



major axis: $(\pm 3, 0)$
minor axis: $(0, \pm 2)$

1.5 Functions and Logarithms

A Function: assigns a single value in its range to each point in its domain. Vertical Line Test

A Function is said to be a one-to-one function if every point in its domain has a unique value in its range. Horizontal Line Test

Inverse functions: since each output of a <u>one-to-one</u> function comes from one input, a one-to-one function will have an Inverse.

If $f(x)$ is a function, then $f^{-1}(x)$ is the inverse. $f^{-1}(x) \neq \frac{1}{f(x)}$

<u>Finding Inverse Function</u>

1.) Solve the equation for x

2) Interchange $x + y$. The resulting equation is $f^{-1}(x)$

Example: Find $f^{-1}(x)$ for the following:

$y = 3x - 5$

$3x = y + 5$

$x = \frac{1}{3}y + \frac{5}{3}$

$y = \frac{1}{3}x + \frac{5}{3}$

$\therefore \boxed{f^{-1}(x) = \frac{1}{3}x + \frac{5}{3}}$

Verify that both composites are the identity function

$f(x) = 3x - 5 \qquad f^{-1}(x) = \frac{1}{3}x + \frac{5}{3}$

$f(f^{-1}(x)) = 3(\frac{1}{3}x + \frac{5}{3}) - 5$

$= x + 5 - 5$

$\boxed{f(f^{-1}(x)) = x}$

$f^{-1}(f(x)) = \frac{1}{3}(3x - 5) + \frac{5}{3}$

$= x - \frac{5}{3} + \frac{5}{3}$

$\boxed{f^{-1}(f(x)) = x}$

Logarithm Function

$y = \log_a x$ is the inverse of $y = a^x$

$y = \ln(x) \rightarrow$ Natural logarithm

$y = \log(x) \rightarrow$ Common logarithm

Inverse Properties of logarithms

1.) Base "a": $a^{\log_a(x)} = x$, $\log_a a^x = x$

2.) Base "e": $e^{\ln(x)} = x$, $\ln e^x = x$

Examples of using the Inverse Properties

a.) $\ln x = 4t + 5$

$e^{\ln x} = e^{4t+5}$

$\therefore \boxed{x = e^{4t+5}}$

b.) $e^{3x} = 15$

$\ln e^{3x} = \ln 15$

$\therefore 3x = \ln 15$

$3x \doteq 2.7081$

$\boxed{x = .9027}$

Properties of logs

1.) Product: $\log_a(xy) = \log_a x + \log_a y$

2.) Quotient: $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

3.) Power: $\log_a (x)^y = y \log_a x$

Change log Base Formula

$$\log_a x = \frac{\ln x}{\ln a}, \quad a \neq 1, a > 0$$

$$f(x) = \log_5 x = \frac{\ln x}{\ln 5}$$

Example:

Sarah invests \$1000.00 in an account that earns 5.25% interest compounded annually. How long will it take the account to reach \$2500.00?

$y = y_0 a^t$ Remember this equation from section 3, where:

y = final amount
y_0 = initial amount
a = 1 plus interest
t = time in years

$$2500 = 1000(1.0525)^t$$
$$2.5 = (1.0525)^t$$

$$\ln 2.5 = \ln (1.0525)^t$$
$$\ln 2.5 = t \ln 1.0525$$

$$t = \frac{\ln 2.5}{\ln 1.0525}$$

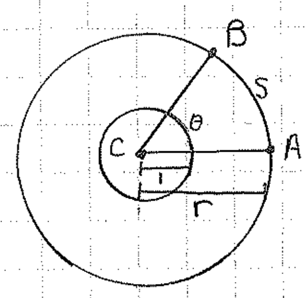
$$t = \frac{.9163}{.0512}$$

$$\boxed{t \doteq 17.9 \text{ years}}$$

Assignment: Page 43 (1-6, 13, 15, 20, 33, 34)

1.6 Trigonometric Functions

Radian Measure (of the angle)



Radian measure is the arc length (s) divided by the radius (r). In other words, it is the ratio of s/r.

Example: $\theta = \frac{s}{r}$

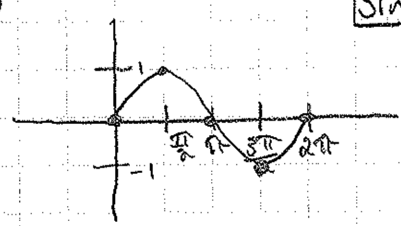
angle(θ)	Radius(r)	Arc length (s)
$\frac{2\pi}{3}$	3	? $\left[\frac{2\pi}{3} \times 3 = 2\pi\right]$
$\frac{\pi}{2}$	? $\left[\frac{\pi}{2} = \frac{2\pi}{r},\ r = 4\right]$	2π

Trig Function Graphs

$y = \sin(x)$

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin(x)$	0	1	0	-1	0

Period: $\frac{2\pi}{1}$
Int: $\frac{2\pi}{4} = \frac{\pi}{2}$
Amp: 1



General Function: $y = A\sin\left[\frac{2\pi}{B}(x - C)\right] + D$

$A = |\text{amplitude}|$
$B = |B|$ is the period
$C =$ Horizontal Shift
$D =$ Vertical Shift

In class, graph: $y = 3\sin\left(\frac{1}{2}x + \pi\right) + 1$

$y = 3\sin\frac{1}{2}(x + 2\pi) + 1$

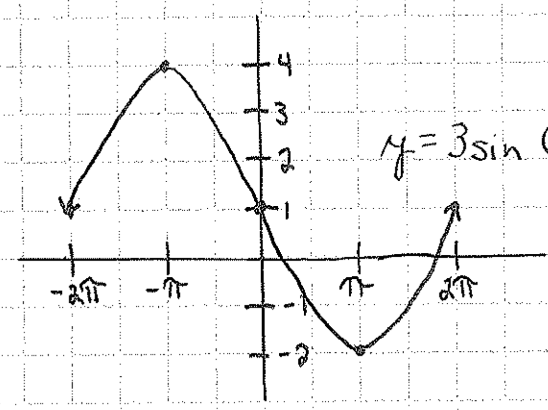
This indicates a horizontal shift of -2π

x	-2π	$-\pi$	0	π	2π
$\frac{1}{2}x$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π
$\frac{1}{2}x + \pi$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin(\frac{1}{2}x + \pi)$	0	1	0	-1	0
$3\sin(\frac{1}{2}x + \pi)$	0	3	0	-3	0
$3\sin(\frac{1}{2}x + \pi) + 1$	1	4	1	-2	1

Amp: 3

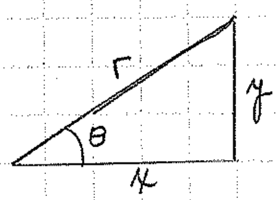
Period: $\frac{2\pi}{1/2} = 4\pi$

Int: $\frac{P}{4} = \pi$



$y = 3\sin\left(\frac{1}{2}x + \pi\right) + 1$

Domain: $(-\infty, \infty)$
Range: $[-2, 4]$



$\tan\theta = \frac{y}{x}$ $\qquad \cos\theta = \frac{x}{r}$

$\sin\theta = \frac{y}{r}$

Do a few examples for finding the values.

Example: $x = 15$ ft, $\theta = 25°$, Find y

$$\tan(25) = \frac{y}{15}$$
$$y = 15\tan(25°)$$
$$y = (15)(.4663)$$
$$\boxed{y = 6.9946 \text{ ft}}$$

Assignment: Page 51 (1-4, 11-14)

Chapter 2

2.1 - Rates of Change and Limits

Average rate of change $= \frac{\Delta y}{\Delta t} = \frac{f(t+h)-f(t)}{h}$

General Equation for freefall: $S = S_0 + V_0 t + \frac{1}{2}at^2$

English: $s = ft$, $t = sec$, $V = ft/sec$, $a = 32\,ft/sec^2$
MKS: $s =$ meters, $t = sec$, $V = m/sec$, $a = 9.8\,m/sec^2$
CGS: $s = cm$, $t = sec$, $V = cm/sec$, $a = 980\,cm/sec^2$

Example: A rock is thrown upward from ground level at a rate of 144 ft/sec.
a.) Find the rock's average velocity during the first 2 sec.
b.) When will the rock hit the ground?

$$S = S_0 + V_0 t + \tfrac{1}{2}at^2$$
$$\boxed{S = 144t - 16t^2}$$

a.) Average Velocity $= \frac{\Delta S}{\Delta t} = \frac{224-0}{2-0} = \boxed{112\,ft/sec}$

side work
at $t=0 \Rightarrow S = 144(0) - 16(0)^2 = 0\,ft$
at $t=2 \Rightarrow S = 144(2) - 16(4) = 224\,ft$

b.) It will hit the ground when $S=0$, therefore

$$0 = 144t - 16t^2$$
$$16t(9-t) = 0$$
$$16t = 0 \text{ or } 9-t = 0$$
$$\boxed{t=0} \text{ or } \boxed{t = 9sec}$$

(Note: "Speeding" ticket story)

Instantaneous Velocity - velocity at a given instant. Problem is that time "seems" to go to zero.

How could we "get an idea" of the instantaneous velocity at $t = 2sec$ in the example above. (Do this in class - real answer is 80ft/sec)

Limit: The value a function reaches when its variable approaches a given value.

Note: The limit only exists if it approaches the same value from both sides.

Right-handed limit - approaches a value from the positive side

$$\lim_{x \to c^+} f(x)$$

Left-handed limit - approaches a value from the negative side

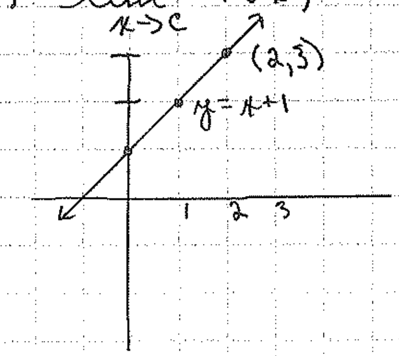
$$\lim_{x \to c^-} f(x)$$

Two-sided limit (or simply the limit) - $\lim_{x \to c} f(x)$

Example: Find a.) $\lim_{x \to 2^+} (x+1)$

b.) $\lim_{x \to 2^-} (x+1)$

c.) $\lim_{x \to 2} (x+1)$



Example 2: $\lim_{x \to 0} \frac{1}{x}$

Put the "Properties of Limits" on pages 61+62 in notebooks.

Trig limits

$$\lim_{\theta \to 0} \sin(\theta) = 0$$

$$\lim_{\theta \to 0} \cos(\theta) = 1$$

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$$

Example: $\lim_{\theta \to 0} \frac{3\tan(\theta)}{\theta} = 3\lim_{\theta \to 0} \frac{\tan(\theta)}{\theta} = 3\lim_{\theta \to 0} \left(\frac{\sin(\theta)}{(\theta)}\right)\left(\frac{1}{\cos(\theta)}\right)$

$= 3 \cdot 1 \cdot 1$

$= \boxed{3}$

(Hand out the "limit" sheet)

Limits

Limits are one of the easiest Calculus problems to do (at least to this point)! You have only 2 types. They are:

Chapter 2 Section 1

1.) The limit of a function as its variable goes to a specific number. In this case, you have 3 steps to follow.

a. Put the specific number into the expression to see if you get an answer. If you get a specific answer such as 4, then you are done and you have the answer. If you get something that is undefined, positive infinity, or negative infinity; you must try step b.
b. At this point, you must try to factor the expression (both numerator and denominator). If you are lucky, it will factor and you can then substitute the value into the new expression and get your answer. If it does not factor easily, you must try long division on the problem to find the factors. Once this is done, you can substitute the vlue into the expression and get your answer.
c. If both of these fail, you must try the right and left handed limit to see if they converge at a given value. If they converge at the same value, then that is the limit. If not, then the limit Does Not Exist (DNE)!

Chapter 2 Section 2

2.) The limit of a function as its variable goes to infinity. In this case you will simply divide the numerator and the denominator by the variable with the highest exponent in the denominator. Once you have done this correctly, use the definition that Lim(1/X) = 0 as X goes to infinity. This should give you the answer.

Sandwich or Squeeze Theorem:

If $g(x) \le f(x) \le h(x)$ for all $x \ne C$ in some interval about C, and

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$$

then $\lim_{x \to c} f(x) = L$

Examples:

If $3x \le f(x) \le x^3 + 2$ for $0 \le x \le 2$, evaluate

$$\lim_{x \to 1} f(x)$$

$$\lim_{x \to 1} 3x = 3 \qquad \lim_{x \to 1} x^3 + 2 = 3$$

$$\boxed{\therefore \lim_{x \to 1} f(x) = 3}$$

$$\lim_{x \to 0} x^2 \cos\left(\frac{1}{x^2}\right) =$$

$$-1 \le \cos\left(\frac{1}{x^2}\right) \le 1$$

$$-x^2 \le x^2 \cos\left(\frac{1}{x^2}\right) \le x^2$$

$$\lim_{x \to 0} (-x^2) \le \lim_{x \to 0} \left(x^2 \cos\left(\frac{1}{x^2}\right)\right) \le \lim_{x \to 0} x^2$$

$$0 \le \lim_{x \to 0} \left(x^2 \cos\left(\frac{1}{x^2}\right)\right) \le 0$$

$$\boxed{\therefore \lim_{x \to 0} \left(x^2 \cos\left(\frac{1}{x^2}\right)\right) = 0}$$

Find $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$

Assignment: Page 66 (1-3, 25-34, 44, 58)

2.2 - Limits Involving Infinity

In this section we will use all the "tricks" you learned in the previous section.

One additional trick is to divide the num. and den. by the same thing.

Also that $\lim_{x \to \infty} \frac{1}{x} = 0$

Example: $\lim_{x \to \infty} \frac{3-2x^4}{x+1}$

$$\lim_{x \to \infty} \frac{4x^2-x}{2x^3-5}$$

$$\lim_{x \to \infty} \frac{5x^4+4x}{6x^4+3x^2+2}$$

Note: $\frac{m}{m_0} = \frac{1}{\sqrt{1-\frac{V^2}{V_c^2}}}$

Assignment: Page 76 (9-12, 13-20, 21-26)

2.3 Continuity

A function is said to be continuous if the graph contains no breaks, no jumps, no holes, and no asymptotes.

The following condictions must be true:

1.) $f(c)$ is defined

2) $\lim_{x \to c} f(x)$ exists

3) $\lim_{x \to c} f(x) = f(c)$

Types of Discontinous Functions

1.) Removable discontinuity: $f(x)$ can be made continous by defining $f(x)$ at $x = c$. This would be a hole in the graph. (Occurs when $f(c) = \frac{0}{0}$)

2.) Non-Removable discontinuity: $f(x)$ can not be defined.

a.) Asymptote ($\lim_{x \to c} f(x) = \frac{N}{0}$)

b.) Jump or Break ($\lim_{x \to c} f(x) = DNE$)

Examples:

1.) Find all points of discontinuity for $f(x) = \frac{x^3-1}{x-1}$

2.) Determine if $f(x) = x^2$ is continous at $x = 1$

3.) Find the points of discontinuity for $f(x) = \frac{x-1}{x^2-x}$

4.) Find any discontinuity for

$$f(x) = \begin{cases} -2x, & x \leq 2 \\ x^2-4x+1, & x > 2 \end{cases}$$

Assignment: Page 84 (1, 2, 14)

2.4 Rates of Change and Tangent Lines

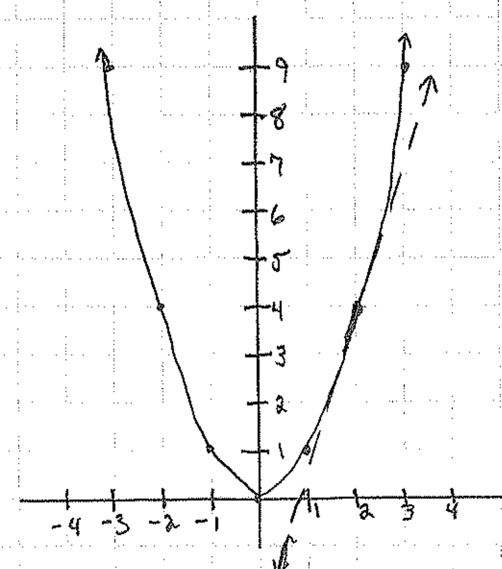
The tangent line is a rate of change and determines the direction of a body's motion at every moment along its path.

The problem is "How do we find the slope of the tangent line"? Pierre de Fermat "invented" the dynamic approach in 1629 to solve this problem.

We are going to solve this problem today!

Find the slope of the line tangent to the curve $y = x^2$ at the point $P(2,4)$, and then find the equation for the tangent line.
(Note: Slope of the Secant line = Average Rate of Change
Slope of the Tangent line = Instanous Rate of Change)

x	y
0	0
1	1
2	4
3	6
-1	1
$\frac{3}{2}$	$\frac{9}{4}$



$(2,4)$ + $(1,1)$

$m = \frac{\Delta y}{\Delta x} = \frac{4-1}{2-1} = 3$

$(2,4)$ + $(\frac{3}{2}, \frac{9}{4})$

$m = \frac{\Delta y}{\Delta x} = \frac{4-\frac{9}{4}}{2-\frac{3}{2}} = 3.5$

$$m_{secant} = \frac{f(x+h)-f(x)}{(x+h)-x} = \frac{f(x+h)-f(x)}{h}$$

$$\therefore m_{tan} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Given: $f(x) = x^2$, $P(2,4)$

$$m_{sec} = \frac{f(2+h)-f(2)}{(2+h)-2} = \frac{(2+h)^2-2^2}{2+h-2} = \frac{4+4h+h^2-4}{h} = \frac{4h+h^2}{h} = \boxed{4+h}$$

$$m_{tan} = \lim_{h\to 0}(4+h) = \boxed{4}$$

Equation of the tangent line at $P(2,4)$

$m = 4,\ P(2,4)$

$y - y_1 = m(x - x_1)$

$y - 4 = 4(x-2)$

$y - 4 = 4x - 8$

$\boxed{y = 4x - 4}$

(Note: The normal line to a curve at a point is the line perpendicular to the tangent line at that point.)

Find the normal to $y = 4x - 4$ at $P(2,4)$

$m_N = -\frac{1}{4}\quad P(2,4)$

$y - y_1 = m(x - x_1)$

$y - 4 = -\frac{1}{4}(x-2)$

$y - 4 = -\frac{1}{4}x + \frac{1}{2}$

$\boxed{y = -\frac{1}{4}x + \frac{9}{2}}$

Assignment: Page 92 (1, 2, 10-12)

3.1 Derivative of a Function

Derivative: Rate of change of a function at any given point. This is the Instantanous rate of change for the function.

This is Known as Differential Calculus!

Symbols used for derivatives: $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

y', $f'(x)$, $\frac{dy}{dx}$

Definition of a Derivative (Delta Process) - 4 steps

Given: $y = f(x)$

1.) Determine $f(x+h)$
2.) Calculate $f(x+h) - f(x)$
3.) Divide by h
4.) Take the limit as $h \to 0$ $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Find $f'(x)$ for $f(x) = x^2$

1.) $f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$
2.) $f(x+h) - f(x) = x^2 + 2xh + h^2 - x^2$
$= 2xh + h^2$
3.) $\frac{f(x+h)-f(x)}{h} = \frac{2xh + h^2}{h} = 2x + h$
4.) $\lim_{h \to 0} (2x + h) = \boxed{2x}$

Cover the following form: $f(x) = \frac{1}{x^2}$, $f(x) = \sqrt{2x+1}$, $f(x) = \frac{1}{\sqrt{2x+1}}$

Assignment: Page 105 (5-16)

3.2 Differentiability

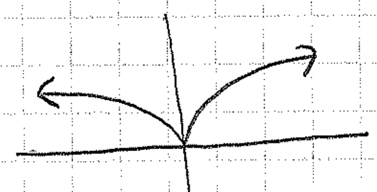
$f'(a)$ may fail to exist if the limit of the secant line, $\dfrac{f(x)-f(a)}{x-a}$ fails to approach a limit as $x \to a$.

Examples would be:

1.) A corner

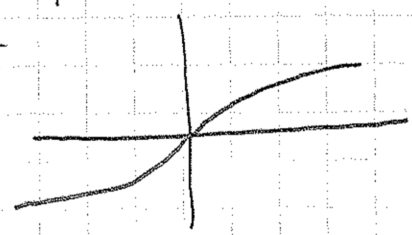
$y = |x|$

2.) A cusp



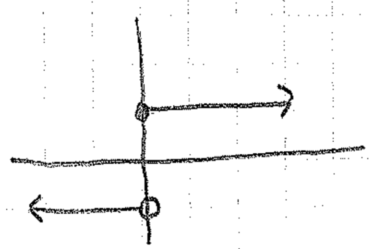
$y = x^{2/3}$

3.) A vertical tangent



$y = \sqrt[3]{x}$

4.) A discontinuity



$$y = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

Theorem: Differentiability Implies Continuity. If $f(x)$ has a derivative at $x=a$, then $f(x)$ is continuous at $x=a$.

Theorem: (IVT) Intermediate Value Theorem. If a and b are any two points in an interval on which $f(x)$ is differentiable, then $f'(x)$ takes on every value between $f'(a)$ and $f'(b)$.

Assignment: Page 114 (5-10)

3.3 Rules for Differentiation

1.) Derivative of a Constant: $\frac{d(c)}{dx} = 0$, c is any constant

2.) Simple Power Rule: $\frac{d(x^n)}{dx} = nx^{n-1}$

3.) Constant Multiple Rule: $\frac{d(cf(x))}{dx} = cf'(x)$

4.) Sum and Difference Rule: $\frac{d(u \pm v)}{dx} = \frac{d(u)}{dx} + \frac{d(v)}{dx}$

Do a few examples: $f(x) = x+3$
$f(x) = x^2+4$

Finding Horizontal Tangents: Answer the question - Why are horizontal tangents important?

Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

5.) Simple Product Rule: $\frac{d(u \cdot v)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

Find $f'(x)$ for $f(x) = (x^3+1)(x^2+5)$

6.) Simple Quotient Rule: $\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Find $f'(x)$ for $f(x) = \frac{x^3+1}{x^2+5}$

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ of $y = (3x+1)^3$

Note: Review the use of Pascal's Triangle to expand.

Assignment: Page 124 (1-23 odd)

3.4 Velocity and Other Rates of Change

Velocity is only one type of rate. Other examples are in the medical field, engineering, economics, and other fields.

The Rate of Change (Instantaneous) is the derivative with respect to a variable.

Example: a.) (Oil Slick) Find the rate of change of the area of a circle with respect to the radius.

$$A = \pi r^2$$

b.) Evaluate $\frac{dA}{dr}$ at $r = 5$, $r = 10$.

A rock is thrown upward from the ground with a velocity of 160 ft/sec. It reaches a height of $s = 160t - 16t^2$ after t sec.

a.) How high does the rock go?

b.) What is the velocity of the rock when it is 256 ft above the ground?

c.) What is the acceleration of the rock at any time t.

d.) When does the rock hit the ground?

Assignment: Page 135 (1, 2, 8, 13–16)

3.6 Derivatives of Trigonometric Functions

You must Know these formulas:

1.) $\frac{d(\sin(x))}{dx} = \cos(x)$

2.) $\frac{d(\cos(x))}{dx} = -\sin(x)$

3.) $\frac{d(\tan(x))}{dx} = \sec^2(x)$

4.) $\frac{d(\sec(x))}{dx} = \sec(x)\tan(x)$

5.) $\frac{d(\cot(x))}{dx} = -\csc^2(x)$

6.) $\frac{d(\csc(x))}{dx} = -\csc(x)\cot(x)$

(Note: Give sin + cos, and then calculate the others in class)

Look at "Simple Harmonic Motion" on page 143.

Assignment: Page 146 (1-10, find $\frac{dy}{dx}$)
146 (13-16, part b only)

4.1 Chain Rule

Given: $y = 4x^6 + 4x^3 + 1$

$$y' = \frac{dy}{dx} = 24x^5 + 12x^2$$
$$= 12x^2(2x^3+1)$$

Looking at a simple problem: $y = 8x + 4 = 4(2x+1)$

Let $u = 2x+1$

then $y = 4u$

$$y = 8x + 4 \qquad\qquad \frac{dy}{du} = 4, \quad \frac{du}{dx} = 2$$

$$\frac{dy}{dx} = 8 \qquad\qquad \frac{dy}{du} \cdot \frac{du}{dx} = 4 \cdot 2 = 8$$

Now, let us look at the first problem again:

$$y = 4x^6 + 4x^3 + 1 = (2x^3+1)^2$$

Let $u = 2x^3 + 1$

then $y = u^2$

$$\frac{dy}{du} = 2u, \quad \frac{du}{dx} = 6x^2$$

$$\frac{dy}{du} \cdot \frac{du}{dx} = 2u \cdot 6x^2$$
$$= 12x^2 u$$
$$= 12x^2(2x^3+1)$$

Restatement of the rules of derivatives using the Chain Rule:

$u = f(x)$

1.) General Power Rule: $y = u^n$

$$\frac{dy}{dx} = n u^{n-1} \frac{du}{dx}$$

2.) General Product Rule: $y = u \cdot V$

$$\frac{dy}{dx} = u\frac{dV}{dx} + V\frac{du}{dx}$$

Example: $y = (x^2+1)(2x^3+5)$

Method #1: (Expand)

$$y = 2x^5 + 5x^2 + 2x^3 + 5$$
$$y = 2x^5 + 2x^3 + 5x^2 + 5$$
$$\boxed{\frac{dy}{dx} = 10x^4 + 6x^2 + 10x}$$

Method #2: General Product Rule

$$y = (x^2+1)(2x^3+5)$$

Let $u = x^2+1$ Let $V = 2x^3+5$

$\frac{du}{dx} = 2x$ $\frac{dV}{dx} = 6x^2$

$$\frac{dy}{dx} = u\frac{dV}{dx} + V\frac{du}{dx}$$
$$\frac{dy}{dx} = (x^2+1)(6x^2) + (2x^3+5)(2x)$$
$$\frac{dy}{dx} = 6x^4 + 6x^2 + 4x^4 + 10x$$
$$\boxed{\frac{dy}{dx} = 10x^4 + 6x^2 + 10x}$$

3.) General Quotient Rule: $y = \frac{u}{V}$

$$\frac{dy}{dx} = \frac{V\frac{du}{dx} - u\frac{dV}{dx}}{V^2}$$

Application Example:

Suppose that the temperature T of food placed in a refrigerator drops according to the equation

$$T = 10\left(\frac{4t^2+16t+75}{t^2+4t+10}\right)$$

where t is the time in hours. What is the initial temperature of the food? Find the rate of change of T with respect to t when:

a.) $t = 1$ hour

b.) $t = 3$ hours

c.) $t = 5$ hours

d.) $t = 10$ hours

Answers: at $t=0$, $T = 75°$

a) at $t=1$, $\frac{dT}{dt} = -9.33°/\text{hr}$

b.) at $t=3$, $\frac{dT}{dt} = -3.64°/\text{hr}$

c.) at $t=5$, $\frac{dT}{dt} = -1.61°/\text{hr}$

d.) at $t=10$, $\frac{dT}{dt} = -.37°/\text{hr}$

What is the limit of T as $t \to \infty$?

$$\lim_{t\to\infty} 10\left(\frac{4t^2+16t+75}{t^2+4t+10}\right) = 40°$$

Example: Find $\frac{dy}{dx}$ if $y = \tan(3-\cos(2x))$

$$\frac{dy}{dx} = \sec^2(3-\cos(2x))\,(\sin(2x))\,(2)$$

$$\boxed{\text{Note: } \frac{dy}{dx} = \frac{d(\tan(3-\cos(2x))}{dx} \cdot \frac{d(3-\cos(2x))}{dx} \cdot \frac{d(2x)}{dx}}$$

$$\therefore \frac{dy}{dx} = 2\sin(2x)\sec^2(3-\cos(2x))$$

Parametric Equation

A parametric equation is a pair of equations that define the motion of an object in terms of a third variable.

Find the equation of the line tangent to the curve defined by

$$x = \sec(t), \quad y = \tan(t)$$

where $t = \pi/4$

$$\frac{dx}{dt} = \sec(t)\tan(t) \qquad \frac{dy}{dt} = \sec^2(t)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sec^2(t)}{\sec(t)\tan(t)}$$

$$= \frac{\sec(t)}{\tan(t)}$$

$$= \frac{\frac{1}{\cos(t)}}{\frac{\sin(t)}{\cos(t)}}$$

$$= \frac{1}{\cos(t)} \cdot \frac{\cos(t)}{\sin(t)}$$

$$= \frac{1}{\sin(t)}$$

$$\frac{dy}{dx} = m = \frac{1}{\sin(t)}, \text{ at } t = \frac{\pi}{4}, \ m = \frac{1}{\frac{\sqrt{2}}{2}} = \sqrt{2} \doteq 1.4142$$

$$\left.\begin{array}{l} \text{at } \frac{\pi}{4}, \ x = \sec(\frac{\pi}{4}) = \sqrt{2} \doteq 1.4142 \\ \text{at } \frac{\pi}{4}, \ y = \tan(\frac{\pi}{4}) = 1 \end{array}\right\} \Rightarrow (x, y) = (\sqrt{2}, 1)$$

$$\therefore \quad y - y_1 = m(x - x_1)$$

$$y - 1 = \sqrt{2}(x - \sqrt{2})$$

$$y - 1 = \sqrt{2}x - 2$$

$$\boxed{y = \sqrt{2}x - 1}$$

Worksheet Problems

1.) $y=(2x^3-5x^2+4)^5$

2.) $y=(x^2+3)^3$

3.) $y=(x^3-4)^5$

4.) $y=2\sqrt[5]{x^3}$

5.) $y=\sqrt{2x^3-4x+5}$

6.) $y=(x^2+1)^3(x^3-1)^2$

7.) $y=(x+1)^3(2x-1)^{4/3}$

8.) $y=\dfrac{x^3}{\sqrt[3]{3x^2-1}}$

9.) $y=\dfrac{(x^2+5)^4}{(x^3+2)^7}$

10.) $y=\dfrac{(3x^2-4)^5}{(2x^3+1)^{5/7}}$

Assignment: P158 (1-8)

P158 (13-31) odd

P158 (41-47) odd

4.2 Implicit Differentiation

Implicit Differentiation - Differentiation when one variable is not given explicitly as a function of the other variable.

<u>Rules for Implicit Differentiation</u>

1.) Differentiate both sides of the equation with respect to x.

2.) Collect all terms involving $\frac{dy}{dx}$ on the left side of the equation, and move all the others to the right side of the equation.

3.) Factor $\frac{dy}{dx}$ out of the left side of the equation.

4.) Solve for $\frac{dy}{dx}$

Example: Find $\frac{d^2y}{dx^2}$ if $2x^4-3y^3=6$

Solution:
$$8x^3-9y^2\frac{dy}{dx}=0$$

$$\boxed{\frac{dy}{dx}=\frac{8x^3}{9y^2}}$$

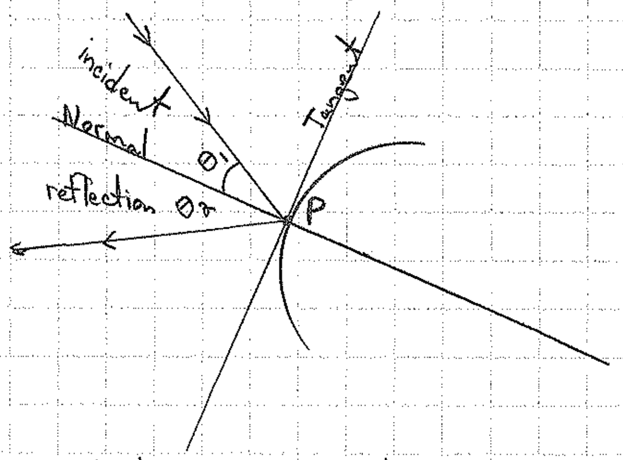
$$\frac{d^2y}{dx^2}=\frac{(9y^2)(24x^2)-(8x^3)(18y\frac{dy}{dx})}{81y^4}$$

$$=\frac{216x^2y^2-144x^3y\left(\frac{8x^3}{9y^2}\right)}{81y^4}$$

$$=\frac{216x^2y^2-128x^6y^{-1}}{81y^4}$$

$$=\boxed{\frac{8x^2(27y^3-16x^4)}{81y^5}}$$

Mirrors, Lenses, Tangents, and Normal Lines



Normal Line - the line perpendicular to the tangent line of a curve at a given point P.

Find the tangent and normal to the ellipse

$$x^2 - xy + y^2 = 7 \text{ at } (-1,2)$$

$$2x - (x\frac{dy}{dx} + y) + 2y\frac{dy}{dx} = 0$$

$$2x - x\frac{dy}{dx} - y + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} - x\frac{dy}{dx} = y - 2x$$

$$\boxed{\frac{dy}{dx} = \frac{y-2x}{2y-x}}$$

at $(-1,2)$, $\frac{dy}{dx} = m = \frac{2-2(-1)}{2(2)-(-1)} = \frac{4}{5}$

<u>Tangent</u>: $y - y_1 = m(x - x_1)$

$$y - 2 = \frac{4}{5}(x-(-1))$$

$$y - 2 = \frac{4}{5}x + \frac{4}{5}$$

$$\boxed{y = \frac{4}{5}x + \frac{14}{5}}$$

Normal: $m = -\frac{5}{4}$

$$y - 2 = -\frac{5}{4}(x-(-1))$$

$$y - 2 = -\frac{5}{4}x - \frac{5}{4}$$

$$\boxed{y = -\frac{5}{4}x + \frac{3}{4}}$$

Assignment: Page 167 (1-19) odd

4.3 Derivatives of Inverse Trigonometric Functions

$$y = \sin^{-1}(x) \qquad \left(-\frac{\pi}{2} < y < \frac{\pi}{2}\right)$$

$$\sin(y) = x$$

$$\cos(y)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos(y)} \Rightarrow \frac{1}{\sqrt{1-\sin^2(y)}} \Rightarrow \frac{1}{\sqrt{1-x^2}}$$

The 6 Inverse Trig Derivatives

1.) $$\frac{d(\sin^{-1}(u))}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

2.) $$\frac{d(\cos^{-1}(u))}{dx} = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$$

3.) $$\frac{d(\tan^{-1}(u))}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$$

4.) $$\frac{d(\csc^{-1}(u))}{dx} = \frac{-1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$

5.) $$\frac{d(\sec^{-1}(u))}{dx} = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$

6.) $$\frac{d(\cot^{-1}(u))}{dx} = \frac{-1}{1+u^2}\frac{du}{dx}$$

Example: P175(7): $y = x\sin^{-1}(x) + \sqrt{1-x^2}$

$$\frac{dy}{dx} = x\left[\frac{1}{\sqrt{1-x^2}}\right] + \sin^{-1}(x)[1] + \frac{1}{2}(1-x^2)^{-1/2}(-2x)$$

$$= \frac{x}{\sqrt{1-x^2}} + \sin^{-1}(x) - \frac{x}{\sqrt{1-x^2}}$$

$$= \boxed{\sin^{-1}(x)}$$

Assignment: Page 175 (2-8 even) (13-21 odd)

Derivatives, Integrals, and Properties
Of Inverse Trigonometric Functions and Hyperbolic Functions

(On this handout, a represents a constant, u and x represent variable quantities)

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad (|u|<1)$$

$$\frac{d}{dx}\cos^{-1}u = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad (|u|<1)$$

$$\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx}$$

$$\frac{d}{dx}\csc^{-1}u = \frac{-1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad (|u|>1)$$

$$\frac{d}{dx}\sec^{-1}u = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad (|u|>1)$$

$$\frac{d}{dx}\cot^{-1}u = \frac{-1}{1+u^2}\frac{du}{dx}$$

Identities for Hyperbolic Functions

$$\sinh 2x = 2\sinh x\cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$

$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

Integrals Involving Inverse Trigonometric Functions

$$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right) + C \qquad (\text{Valid for } u^2 < a^2)$$

$$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C \qquad (\text{Valid for all } u)$$

$$\int \frac{1}{u\sqrt{u^2-a^2}}\,du = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C \qquad (\text{Valid for } u^2 > a^2)$$

Derivatives of Hyperbolic Functions

$$\frac{d}{dx}\sinh u = \cosh u\frac{du}{dx}$$

$$\frac{d}{dx}\cosh u = \sinh u\frac{du}{dx}$$

$$\frac{d}{dx}\tanh u = \operatorname{sech}^2 u\frac{du}{dx}$$

$$\frac{d}{dx}\coth u = -\operatorname{csch}^2 u\frac{du}{dx}$$

$$\frac{d}{dx}\operatorname{sech} u = -\operatorname{sech} u\tanh u\frac{du}{dx}$$

$$\frac{d}{dx}\operatorname{csch} u = -\operatorname{csch} u\coth u\frac{du}{dx}$$

The Six Basic Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Inverse Hyperbolic Identities

$$\operatorname{sech}^{-1}x = \cosh^{-1}\left(\frac{1}{x}\right)$$

$$\operatorname{csch}^{-1}x = \sinh^{-1}\left(\frac{1}{x}\right)$$

$$\coth^{-1}x = \tanh^{-1}\left(\frac{1}{x}\right)$$

Integrals of Hyperbolic Functions

$$\int \sinh u \, du = \cosh u + C$$

$$\int \cosh u \, du = \sinh u + C$$

$$\int \operatorname{sech}^2 u \, du = \tanh u + C$$

$$\int \operatorname{csch}^2 u \, du = -\coth u + C$$

$$\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$$

$$\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$$

Integrals Involving Inverse Hyperbolic Functions

$$\int \frac{1}{\sqrt{a^2+u^2}} \, du = \sinh^{-1}\left(\frac{u}{a}\right) + C \qquad (a > 0)$$

$$\int \frac{1}{\sqrt{u^2-a^2}} \, du = \cosh^{-1}\left(\frac{u}{a}\right) + C \qquad (u > a > 0)$$

$$\int \frac{1}{a^2-u^2} \, du = \begin{cases} \dfrac{1}{a}\tanh^{-1}\left(\dfrac{u}{a}\right) + C & (\text{if } u^2 < a^2) \\ \dfrac{1}{a}\coth^{-1}\left(\dfrac{u}{a}\right) + C & (\text{if } u^2 > a^2) \end{cases}$$

$$\int \frac{1}{u\sqrt{a^2-u^2}} \, du = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C \qquad (0 < u < a)$$

$$\int \frac{1}{u\sqrt{a^2+u^2}} \, du = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C$$

Derivatives of Inverse Hyperbolic Functions

$$\frac{d}{dx}\sinh^{-1} u = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}\cosh^{-1} u = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx} \qquad (u > 1)$$

$$\frac{d}{dx}\tanh^{-1} u = \frac{1}{1-u^2}\frac{du}{dx} \qquad (|u| < 1)$$

$$\frac{d}{dx}\operatorname{csch}^{-1} u = \frac{-1}{|u|\sqrt{1+u^2}}\frac{du}{dx} \qquad (u \neq 0)$$

$$\frac{d}{dx}\operatorname{sech}^{-1} u = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx} \qquad (0 < u < 1)$$

$$\frac{d}{dx}\coth^{-1} u = \frac{1}{1-u^2}\frac{du}{dx} \qquad (|u| > 1)$$

Expressing Inverse Hyperbolic Functions As Natural Logarithms

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}) \qquad (-\infty < x < \infty)$$

$$\cosh^{-1} x = \ln(x + \sqrt{x^2-1}) \qquad (x \geq 1)$$

$$\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x} \qquad (|x| < 1)$$

$$\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) \qquad (0 < x \quad 1)$$

$$\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right) \qquad (x \neq 0)$$

$$\coth^{-1} x = \frac{1}{2}\ln\frac{x+1}{x-1} \qquad (|x| > 1)$$

Alternate Form For Integrals Involving Inverse Hyperbolic Functions

$$\int \frac{1}{\sqrt{u^2 \pm a^2}} \, du = \ln(u + \sqrt{u^2 \pm a^2}) + C$$

$$\int \frac{1}{a^2-u^2} \, du = \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + C$$

$$\int \frac{1}{u\sqrt{a^2 \pm u^2}} \, du = -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2 \pm u^2}}{|u|}\right) + C$$

4.4 Derivatives of Exponential and Logarithmic Functions

Derivative of Exponential Function e^x

$$e = 2.71828\ldots$$

$$* \quad \frac{d(e^u)}{dx} = e^u \cdot \frac{du}{dx}$$

Examples: $y = e^{-2x}$ $\quad \frac{dy}{dx} = e^{-2x}(-2)$

$y = e^{x^3}$ $\quad \frac{dy}{dx} = e^{x^3}(3x^2)$

$y = e^{\cos(x)}$ $\quad \frac{dy}{dx} = e^{\cos(x)}(-\sin(x))$

$y = e^{(x+x^2)}$ $\quad \frac{dy}{dx} = e^{(x+x^2)}(1+2x)$

Derivative of a^x

Key: $a^x = e^{x\ln a}$ $\qquad$ (Note: $e^{x\ln a} = e^{\ln a^x} = a^x$)

$$* \quad \frac{d(a^u)}{dx} = a^u \ln a \frac{du}{dx}$$

Example: $y = 2^x - 3$

$$\frac{dy}{dx} = 2^x \ln(2)(1) - 0$$

$$\frac{dy}{dx} = 2^x \ln(2)$$

Question: A what point on the graph of $y = 2^x - 3$ does the tangent line have a slope of 21.

$$21 = 2^x \ln 2$$

$$2^x = \frac{21}{\ln 2}$$

$$\ln 2^x = \ln \frac{21}{\ln 2}$$

$$x \ln 2 = \ln 21 - \ln(\ln 2)$$

$$x = \frac{\ln(21) - \ln(\ln 2)}{\ln(2)}$$

$$x \doteq 4.921$$

$$y = 27.29$$

$$\Rightarrow \boxed{(4.921, 27.29)}$$

Derivative of $\ln(x)$

$$* \quad \frac{d(\ln u)}{dx} = \frac{1}{u} \frac{du}{dx}$$

Example: $y = \ln(x^2+1)$

$$\frac{dy}{dx} = \frac{1}{(x^2+1)} \quad (2x)$$

Properties of logs

1) Product: $\ln(xy) = \ln x + \ln y$
2.) Quotient: $\ln \frac{x}{y} = \ln x - \ln y$
3.) Power: $\ln x^y = y \ln x$

Change base formula

$$\log_a x = \frac{\ln x}{\ln a} \qquad \left[\log_5 x = \frac{\ln x}{\ln 5} \right]$$

Examples: $y = \ln\left(\frac{x}{x^2+1}\right)$

$$y = \ln x - \ln(x^2+1)$$

$$\frac{dy}{dx} = \frac{1}{x} - \frac{2x}{x^2+1}$$

$y = \ln(\cos(4x))$

$$\frac{dy}{dx} = \frac{1}{\cos(4x)} \cdot (-\sin(4x))(4)$$

$$= -4\tan(4x)$$

Find $\frac{dy}{dx}$: $y = \log_{10}(x^2) = \frac{\ln(x^2)}{\ln 10}$

$$\frac{dy}{dx} = \frac{1}{\ln 10} \cdot \frac{1}{x^2}(2x)$$

$$\frac{dy}{dx} = \frac{1}{\ln 10} \cdot \frac{2}{x} = \boxed{\frac{2}{x \ln 10}}$$

$$* \quad \frac{d(\log_a u)}{dx} = \frac{1}{u \ln a} \frac{du}{dx}$$

Assignment: Page 183 (1-27 odd)

5.1 Extreme Values of Functions

Maxima and minima Theory: There are many problems in math that deal with finding the "best" way of doing something. In this section, we will learn how to solve this type of problem.

There are two types of Extreme Values:

1.) Absolute (Global) - This would be the largest (Maximum) or smallest (minimum) value the function will obtain over the Domain of the function

Extreme Value Theorem: If $f(x)$ is continuous on a closed interval $[a,b]$, then $f(x)$ has both a maximum and minimum value on the interval.

2.) Local (Relative) - These occur where $f'(c)=0$

Critical Points - points where $f'(x)=0$ or $f'(x)$ does not exist.

Stationary Points - points where $f'(x)=0$.

Stationary Points can be a maximum, minimum, or inflection point.

Examples: Find all Max/min values
(Note: Need 1st der test)

$f(x) = \frac{1}{\sqrt{4-x^2}}$ (Note: D: $4-x^2>0 \Rightarrow (-2,2)$)

The Method - Given $y = f(x)$

1.) Find all points that may be either a local or absolute max. or min value. This occurs when anyone of these are true:

a.) at the endpoints of the domain. You must always check these values when you have a closed interval.

b.) Find the 1^{st} derivative and set it to zero. The x-value where the 1^{st} der $= 0$ could be the value you are looking for.

c.) Any x-value where the 1^{st} der is undefined.

2.) Test the points to find if they are max or min values. There are 2 ways to test:

a.) 1^{st} der test: 1^{st} der changes $-$ to $+$ $\Rightarrow$ min
1^{st} der changes $+$ to $-$ $\Rightarrow$ max

b.) 2^{nd} der test: $f''(x) > 0 \Rightarrow$ min
$f''(x) < 0 \Rightarrow$ max

Example: $y = 4x^2 - 4x + 1 \quad [0, 1]$

$$y = x^3 - 3x - 2 \quad (-\infty, \infty)$$

Assignment: Page 198 (11-17 odd) (19-29 odd)

5.2 Mean Value Theorem

Note: Michel Rolle (1652-1719) disbelieved in Calculus and tried to disprove it. He is best remembered for his contribution to Calculus.

Rolle's Theorem: If $f(a)=f(b)=0$ and $f(x)$ is continuous on $[a,b]$ and is differentiable on (a,b), then there is at least one number c between $a+b$ where $f'(c)=0$.

* Interesting application of this is in locating solutions for equations.

1.) If $f(x)$ is continuous on $[a,b]$ and is differentiable on (a,b)

2.) $f(a)$ and $f(b)$ have opposite signs, and

3.) $f'(x)$ is never zero between a and b

then $y=f(x)$ has only one zero between $a+b$.

Example: Show that the equation

$$x^4+3x+1=0 \quad , \quad [-2,-1]$$

has exactly one solution on the interval.

Mean Value Theorem: If $y=f(x)$ is continuous on $[a,b]$ and differentiable on (a,b), then

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

Note: MVT really states that somewhere over the interval $[a,b]$ the instan. rate = avg. rate.

Assignment: None

5.3 Connecting f' and f'' with the Graph of $f(x)$

Graphing and How derivatives will help

First Derivative: 1.) Max Value
2.) min Value
3.) Increasing
4.) Decreasing

Second Derivative: 1.) Con-cave Up
2.) Con-cave Down
3.) Points of Inflection

Asymptotes: 1.) Horizontal Asymptotes ($y=b$)

$$\lim_{x\to\infty} f(x) = b \text{ or } \lim_{x\to-\infty} f(x) = b$$

2.) Vertical Asymptotes ($x=a$)

$$\lim_{x\to a^+} f(x) = \pm\infty \text{ or } \lim_{x\to a^-} f(x) = \pm\infty$$

3.) Oblique Asymptotes To find this asymptote, divide the num by the denominator. The resulting linear equation (without the remainder) is the oblique asymptote.

Examples: For the following, find:
a.) Increasing
b.) Decreasing
c.) Con-cave Up
d.) Con-cave Down
e.) Points of Inflection
f.) Max Value
G.) min Valu
H.) Asymptote - Horizontal, Vertical, and Oblique
I.) Graph

Examples:

1.) $y = x^2$

2.) $y = \frac{x^2}{x^2 - 4}$

3.) $y = x - x^2$

4.) $y = x - x^3$

5.) $y = \frac{x}{1+x^2}$

6.) $y = \frac{1}{4}x^4 - \frac{7}{2}x^2 - 6x + 1$

7.) $y = (x-3)^3(x-2)^2$

8.) $y = \frac{x^2}{x-1}$

Assignment: Page 219(1, 2, 3, 5, 7, 8)

5.4 Modeling and Optimization

Strategy for Solving Max/min Problems

1.) Understand the problem. Read the problem and assign variables

2.) Develop a Mathematical Model. Draw a picture, write a primary and secondary equation.

3.) Graph the function. Find the domain (limits) of the function.

4.) Identify the critical points and the endpoints. Find where the derivative is zero.

5.) Solve the Mathematical Model. Check if a Max or min was found, or support your answer with another method.

6.) Interpret the solution. Translate your mathematical result into the problem setting and decide if your answer makes sense. Make sure you answered the questions.

Do examples 3 (page 225) and 4 (page 225) from the book.

Cover "Applications of Max/min Problem-1" from notes.

Do the 6 other examples from my notes.

Assignment: Page 230-236 (give problems close to the sample problems of the day).

Applications of Max/Min Problems - 1

Procedure:

1.) Assign symbols and write down what is known.

2.) Write a "primary" equation for the quantity to be maximized or minimized.

3.) Reduce the "primary" equation to an equation having only one independent variable. Often other "secondary" equations are useful.

4.) Find the max or min by differentiating.

5.) Check to determine if a max or min was found. (Use second der. test or first der. chart if second der. test fails).

6.) Check to make sure you answered the question.

Problems:

1.) What are the dimensions of the rectangle with the largest area and perimeter of 16 cm.

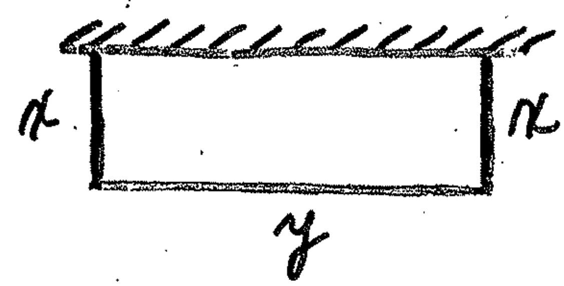
2.) Fred has 100 meters of fencing and wishes to fence in his yard. However, his yard borders a river and needs no fencing on this side. What should be the dimensions of his fenced in yard so that he will have the largest rectangular area possible?

3.) We wish to make an open box out of a square piece of cardboard measuring 12 inches on a side. To do this, we cut out equal squares from the four corners and bend up the four resulting flaps. What size squares should be cut out if we wish the volume of the resulting box to be as large as possible?

4.) Find the positive number for which the sum of its reciprocal and four times its square is the smallest possible.

5.) Find the dimensions of the tin can which will have a volume of 16pi cubic centimeters, and be made of the least amount of material.

Example #1: A farmer has 80 running ft. of fencing available with which to construct a rectanglar enclosure along the side of his barn. What should the dimensions of the rectangle be if the area enclosed is to be a maximum?



$A = xy$

$80 = 2x + y$

$y = 80 - 2x$

$A = x(80 - 2x)$

$\frac{dA}{dx} = 80 - 4x$

$\frac{d^2A}{dx^2} = -4, < 0 \Rightarrow$ max.

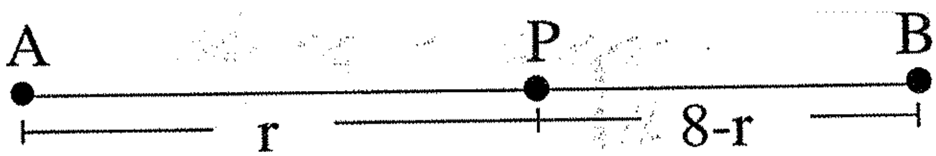
$0 = 80 - 4x$

$4x = 80$

$x = 20$ ft.

$y = 40$ ft.

Example #2: We are given heat sources at points A and B, 8 units apart, with the source at A twice as strong as that at B. If the heat received at a point is inversely proportional to the square of the distance from the heat source and directly proportional to the strength of that source, at what point on the line segment joining A to B will the heat received be a minimum?



$$H_A = \frac{2k}{r^2} \qquad H_B = \frac{k}{(8-r)^2}$$

$$H_T = \frac{k}{(8-r)^2} + \frac{2k}{r^2}$$

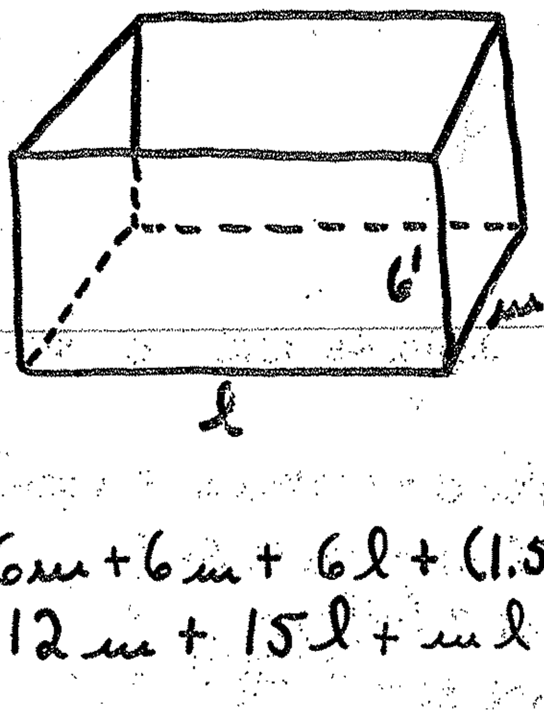
$$\frac{dH}{dr} = \frac{2k}{(8-r)^3} - \frac{4k}{r^3} \qquad \frac{d^2H}{dr^2} = \frac{6k}{(8-r)^4} + \frac{12k}{r^4} > 0 \Rightarrow \text{min}$$

$$0 = \frac{2k}{(8-r)^3} - \frac{4k}{r^3}$$

$$\frac{r^3}{(8-r)^3} = 2$$

$$r(1 + 2^{1/3}) = 8(2^{1/3}) \Rightarrow r = \frac{8(2^{1/3})}{1+(2)^{1/3}} = \frac{10.07}{2.26} = 4.46 \Rightarrow 8-r = 3.54$$

Example #3: An aquarium is to be 6 ft. high and is to have a volume of 750 cu. ft. The base, ends, and back are to be made of slate, but the front is to be made of a plate glass, which costs 1.5 times as much as the slate per sq. ft. What dimensions should be chosen to make the cost of raw materials a minimum?



$V = 750 \text{ ft}^3$

$V = 6 \times l \times w$

$750 = 6 \times l \times w$

$\Rightarrow w = \frac{750}{6l}$

$w = \frac{125}{l}$

$\text{Cost} = 6w + 6w + 6l + (1.5)(6)l + wl$

$= 12w + 15l + wl$

$\text{Cost} = 12\left(\frac{125}{l}\right) + 15l + \left(\frac{125}{l}\right)l$

$= \frac{1500}{l} + 15l + 125$

Example #3 (Page 2)

$$\frac{dc}{d\ell} = -\frac{1500}{\ell^2} + 15$$

$$0 = -\frac{1500}{\ell^2} + 15$$

$$\ell^2 = \frac{1500}{15}$$

$$\ell = 10\text{ ft}$$

$$\frac{d^2c}{d\ell^2} = \frac{3000}{\ell^3} > 0 \Rightarrow \text{min.}$$

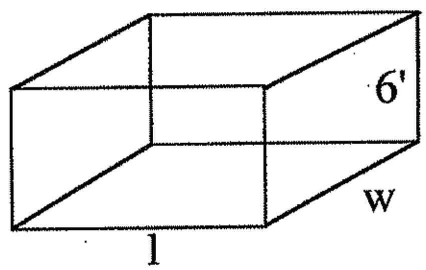
$$750 = 6 \times 10 \times w$$

$$w = \frac{750}{60}$$

$$w = 12.5\text{ ft.}$$

$$\boxed{\begin{aligned} h &= 6' \\ \ell &= 10' \\ w &= 12.5' \end{aligned}}$$

Example #3: An aquarium is to be 6ft. high and is to have a volume of 750 cu. ft. The base, ends, and back are to be made of slate, but the front is to be made of a plate glass, which costs 1.5 times as much as the slate per sq. ft. What dimensions should be chosen to make the cost of raw materials a minimum?



$V=750ft^3$

$V=6lw$

$750=6lw$

$\Rightarrow w=\frac{750}{6l}$

$w=\frac{125}{l}$

$\text{Cost}= 6w+6w+6l+(1.5)(6)l+wl$

$\quad = 12w+15l+wl$

$\text{Cost}= 12(125l^{-1})+15l+(125l^{-1})l$

$\quad = 1500l^{-1}+15l+125$

$$\frac{dC}{dl}= \frac{-1500}{l^2}+15$$

$$0= \frac{-1500}{l^2}+15$$

$$l^2= \frac{1500}{15}$$

$l=10ft$

$$\frac{d^2C}{dl^2}= \frac{3000}{l^3}>0 \Rightarrow \text{min}$$

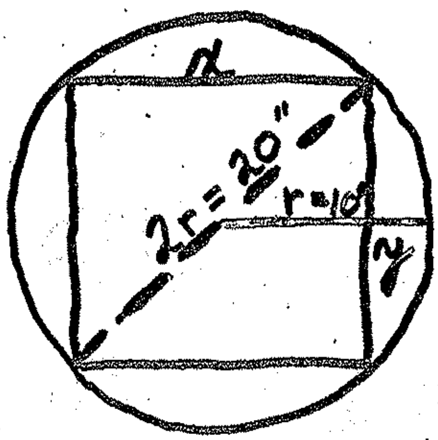
$750= 6(10)(w)$

$$w=\frac{750}{60}$$

$w= 12.5$ ft.

$$\boxed{\begin{array}{l} h= 6' \\ l= 10' \\ w= 12.5' \end{array}}$$

Example #4: Assume that the strength of a rectangular beam varies jointly as the width and the square of the depth. Which rectangular beam cut from a circular log of radius 10 in. will have the maximum strength?



Let x = width
y = depth

$S = kxy^2$
$S = kx(400 - x^2)$

$20^2 = x^2 + y^2$
$y^2 = 20^2 - x^2$
$y^2 = 400 - x^2$

$\frac{dS}{dx} = 400k - 3kx^2$
$0 = 400k - 3kx^2$
$3kx^2 = 400k$
$3x^2 = 400$
$x^2 = \frac{400}{3}$
$x = \sqrt{\frac{400}{3}}$
$x = \frac{20}{\sqrt{3}}$ or $\frac{20\sqrt{3}}{3} = 11.547$

$\frac{d^2S}{dx^2} = -6kx < 0 \Rightarrow$ max.

$400 = \frac{400}{3} + y^2$
$y^2 = 400 - \frac{400}{3}$

$y = \frac{20\sqrt{2}}{\sqrt{3}} = \frac{20\sqrt{6}}{3} = 16.329$

$$\boxed{\begin{array}{l} \text{depth} = \frac{20\sqrt{6}}{3} \\ \text{width} = \frac{20\sqrt{3}}{3} \end{array}}$$

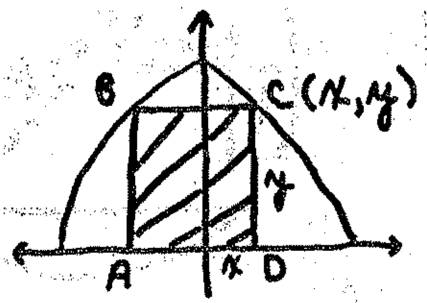
Example #5:

A rectangle ABCD with sides parallel to the coordinate axes is inscribed in the region enclosed by the graph of

$$y = -4x^2 + 4$$

and the x-axis. Find the x and y coordinates of C so that the area of rectangle ABCD is a maximum.

Optional Question: If point C moves along the curve with its x-coordinate increasing at the constant rate of 2 units per second. Find the rate of change of the area of rectangle ABCD when $x = 1/2$



$y = -4x^2 + 4$

$A = 2xy$

$A = 2xy$
$A = 2x(-4x^2+4)$
$A = -8x^3 + 8x$

$\frac{dA}{dx} = -24x^2 + 8$
$0 = -24x^2 + 8$
$24x^2 = 8$
$x^2 = 1/3$
$x = 1/\sqrt{3} = .577$

$y = -4x^2 + 4$
$= -4(.577)^2 + 4$
$y = 2.67$

$$\boxed{(x, y) = (.577, 2.67)}$$

$\frac{d^2A}{dx^2} = -48x, < 0 \Rightarrow \text{max.}$

Optional Question:

$\frac{dx}{dt} = 2$, Find $\frac{dA}{dt}$ when $x = 1/2$

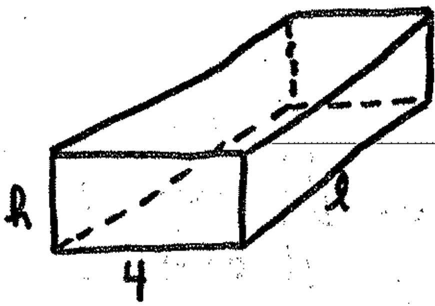
$A = -8x^3 + 8x$
$\frac{dA}{dt} = -24x^2 \frac{dx}{dt} + 8\frac{dx}{dt}$
$= -24(\frac{1}{2})^2(2) + 8(2)$

$= -12 + 16$

$$\boxed{\frac{dA}{dt} = 4 \text{ units}^2/\text{sec}}$$

Example #6:

A tank with a rectangular base and rectangular sides is to be open at the top. It is to be constructed so that its width is 4 meters and its volume is 36 cubic meters. If building the tank costs \$10.00 per square meter for the base and \$5.00 per square meter for the sides, what is the cost of the least expensive tank?



$V = w h l$
$V = 4 h l$
$V = 36$
$36 = 4 h l$
$9 = h l \Rightarrow l = \frac{9}{h}$

$C = 10(4l) + 5(hl + hl + 4h + 4h)$
$C = 40l + 10hl + 40h$

$C = 40\left(\frac{9}{h}\right) + 10h\left(\frac{9}{h}\right) + 40h$
$C = \frac{360}{h} + 90 + 40h$

$\frac{dC}{dh} = -\frac{360}{h^2} + 40$

$\frac{360}{h^2} = 40$

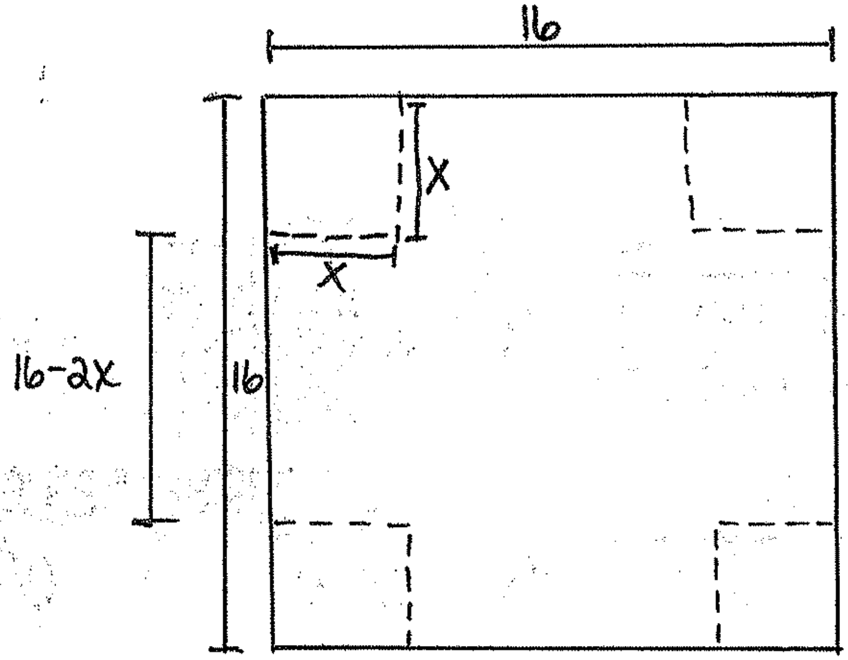
$40h^2 = 360$
$h^2 = 9$
$h = 3$

$l = \frac{9}{3} = 3$

$\frac{d^2C}{dh^2} = \frac{720}{h^3}, > \emptyset \Rightarrow \text{min.}$

$C = 40l + 10hl + 40h$
$C = 40(3) + 10(3)(3) + 40(3)$
$C = 120 + 90 + 120$
$C = 240 + 90$
$\boxed{C = \$330.00}$

Example 7: A square sheet of tin 16 inches on a side is to be used to make an open-top box by cutting a small square of tin from each corner and bending up the sides. How large a square should be cut from each corner to make the box have as large a volume as possible?



$$V = l \cdot w \cdot h$$
$$V = (x)(16-2x)(16-2x)$$
$$V = x(256 - 64x + 4x^2)$$
$$V = 4x^3 - 64x^2 + 256x$$
$$dV/dx = 12x^2 - 128x + 256$$
$$0 = (6x-16)(2x-16)$$

$6x - 16 = 0$ $\quad$ $2x - 16 = 0$

$6x = 16$ $\quad$ $2x = 16$

$x = 2\frac{2}{3} = \boxed{2.667}$ $\quad$ $\cancel{x = 8}$

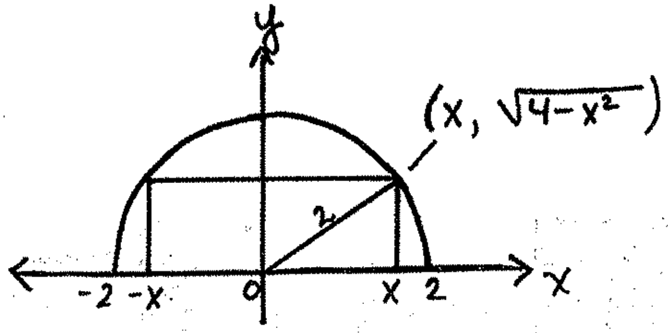
$$\frac{d^2V}{dx^2} = 24x - 128$$

at $x = 2.667$,

$< 0 \Rightarrow$ Max.

Example

A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have and what are its dimensions.



$x^2 + y^2 = 2^2$
$y^2 = 4 - x^2$
$y = \sqrt{4-x^2}$

Length : $2x$

Height : $\sqrt{4-x^2}$

Area: $2x \cdot \sqrt{4-x^2}$

$A = 2x(4-x^2)^{1/2}$

$\frac{dA}{dx} = \frac{-2x^2}{\sqrt{4-x^2}} + 2\sqrt{4-x^2}$

1st derivative chart

$+ \; + \; \underset{\sqrt{2}}{|} \; - \; -$ MAXIMUM

$0 = -2x^2 + 2(4-x^2)$
$0 = -2x^2 + 8 - 2x^2$
$4x^2 = 8$
$x^2 = 2$
$x = \sqrt{2}$

$\text{Area} = 2x \cdot \sqrt{4-x^2}$
$= 2\sqrt{2} \cdot \sqrt{4-2}$
$= 2\sqrt{2} \cdot \sqrt{2}$
$= 2 \cdot 2$
Area $= 4$

Length : $2\sqrt{2}$ Height : $\sqrt{4-2} = \sqrt{2}$

5.5 Linearization and Differentials

Linearization is a way of approximating the value of a function. The idea is to use the tangent line to approximate the value of the function.

Using: $y - f(a) = f'(a)(x-a)$,

this leads to the equation

$$y = f(a) + f'(a)(x-a)$$

which is the linearization formula

$$\boxed{L(x) = f(a) + f'(a)(x-a)}$$

Do example #1 on page 237

Find the linearization of $f(x) = \sqrt{1+x}$ at $x=0$

$$f(x) = (1+x)^{1/2}$$
$$f'(x) = \tfrac{1}{2}(1+x)^{-1/2}(1)$$
$$= \frac{1}{2(1+x)^{1/2}}$$

$\therefore f(0) = 1, \quad f'(0) = \frac{1}{2}$

$$L(x) = 1 + \tfrac{1}{2}(x-0)$$

$$\boxed{L(x) = 1 + \tfrac{1}{2}x}$$

Approxmate $f(1.02) \Rightarrow x = .02$

$\therefore L(.02) = 1 + \frac{1}{2}(.02)$

$\boxed{= 1.01}$

$f(x) = x^4$, $a=1$, find $f(1.01)$

$f'(x) = 4x^3$
$f(1) = 1^4 = 1$
$f'(1) = 4(1)^3 = 4$

$L(x) = 1 + 4(x-1)$
$= 1 + 4x - 4$
$\boxed{= 4x - 3} \Rightarrow \boxed{L(1.01) = 1.04}$

$f(x) = x^2 + 2x$, $a=0$, find $f(.1)$
$f'(x) = 2x+2$
$f(0) = 0$
$f'(0) = 2$

$L(x) = 0 + 2(x-0)$
$= 2x \Rightarrow \boxed{L(.1) = .2}$

Use linearization to find $\sqrt[3]{998}$

$f(x) = \sqrt[3]{x}$, $a = 1000$, find $f(998)$
$f(x) = x^{1/3}$
$f'(x) = \frac{1}{3}x^{-2/3}$

$f(a) = 10$
$f'(a) = \frac{1}{300}$

$L(x) = 10 + \frac{1}{300}(x - 1000)$
$= 10 + \frac{1}{300}x - \frac{10}{3}$
$\boxed{= \frac{x}{300} + \frac{20}{3}}$

$L(998) = \frac{998}{300} + \frac{20}{3}$
$\boxed{\doteq 9.99\overline{3}}$

Finding the differential dy

Given: $y = x^5 + 37x$, $x = 1$, $dx = .01$, find dy

$$\frac{dy}{dx} = 5x^4 + 37$$

$$dy = (5x^4 + 37)\,dx$$

$$dy = (5(1)^4 + 37)(.01) = \boxed{.42}$$

Given: $A = \pi r^2$, $r = 2$, $dr = .02$, find dA

$$\frac{dA}{dr} = 2\pi r$$

$$dA = 2\pi r\,dr = 2\pi(2)(.02) = \boxed{.08\pi}$$

<u>Newton's Method</u> for approximating the zero of a function

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

The steps are:

1.) Given $f(x)$
2.) Find $f'(x)$
3.) Pick a "guess" answer (x_0)
4.) Subs. the value in the equation $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ to get the next value.
5.) Continue until you are satisfied with the answer.

Find the root of: $f(x) = x^3 - 2$ (This is the same as finding the cube root of 2)
use $x_0 = 1$
then $x_1 = 1.\overline{3}$, $x_2 = 1.2638$, $x_3 = 1.2599$

Assignment: Page 246 (1-5, 11, 12, 15-19)

5.6 Related Rates

The rate at which something is changing is simply the derivative of the quantity with respect to time

Strategy for Solving Related Rate Problems

1.) Understand the problem - identify the variable whose rate you seek.

2.) Develop a Mathematical Model - draw a picture and label the parts.

3.) Write an equation relating the variables whose rate you are seeking with the variables whose rate of change you know.

4.) Differentiate both sides of the equation implicitly with respect to time

5.) Substitute values for any quantities that depend on time

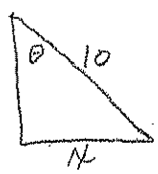
6.) Interpret the solution - answer the question

Assignment: Page 255 (1-35)

Assign the problems that relate to the examples.

related rate

A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 2ft/sec. How fast is the angle between the top of the ladder and the wall changing when the angle is $\frac{\pi}{4}$?



Find $\frac{d\theta}{dt} = ?$ $\frac{dx}{dt} = 2$

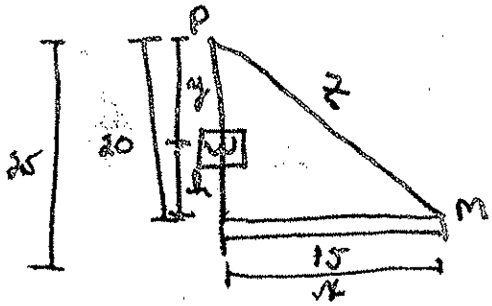
$\sin\theta = \frac{x}{10}$

Related Rates

Ex#1:

A rope running through a pulley at P, bearing a weight W at one end is being pulled at a rate of 6 ft/sec by a man who is holding the other end of the rope in his hand 5 ft above the ground. The pulley is 25 ft above the ground, the rope is 45 ft long, and at a given instant the man is 15 ft away from the ⊥ of the weight. How fast is the weight being raised at this particular instant?

(Find $\frac{dh}{dt}$)



Given: $y+z=45$

$h+y=20$

$20^2+x^2=z^2$

At the given instant: $x=15$ $\quad \frac{dx}{dt}=6$

(at $t=0$)

Key idea: Get an equation that relates x to h

$z=45-y$

$y=20-h$

$\therefore\ z=45-(20-h)$

$\Rightarrow z=25+h$

This yields the equation

$$20^2+x^2=(25+h)^2$$

$$\frac{d(20^2+x^2)}{dt}=\frac{d(25+h)^2}{dt}$$

$$2x\frac{dx}{dt}=2(25+h)\frac{dh}{dt} \Rightarrow \frac{dh}{dt}=\frac{x}{(25+h)}\frac{dx}{dt}$$

subs. $2(15)(6)=2(25+h)\frac{dh}{dt}$

$\frac{dh}{dt}=\frac{(15)(6)}{(25+h)}$

$(20^2+15^2)=(25+h)^2$

$625=(25+h)^2$

$25+h=25$

$$\boxed{\frac{dh}{dt}=\frac{90}{25}=3.6\text{ ft/sec.}}$$

RELATED RATES

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Example #2: A conical icicle, whose height is always 12 times the radius of its base, is being formed by the dripping of water. If the volume is increasing at the rate of 1 cubic cm. per hour, at what rate is the height increasing when the height is 8 cm. ?

FIND: $\frac{dh}{dt}$

GIVEN: $r = \frac{1}{12} \cdot h$

$V = \frac{1}{3} \pi r^2 h$

at a given instant: $\frac{dv}{dt} = 1$; $h = 8$

subs: $V = \frac{1}{3} \pi \left(\frac{1}{12} \cdot h\right)^2 h \Longrightarrow V = \frac{1}{432} \pi h^3$

$\frac{dv}{dt} = \frac{d\left(\frac{1}{432} \pi h^3\right)}{dt}$

$\frac{dv}{dt} = \frac{1}{144} \pi h^2 \frac{dh}{dt} \Longrightarrow \frac{dh}{dt} = \frac{144}{\pi h^2} \frac{dv}{dt}$

subs: $\frac{dh}{dt} = \frac{144}{(3.14)(8)^2}(1)$

$\frac{dh}{dt} = 0.72$ cm/hr.

RELATED RATES

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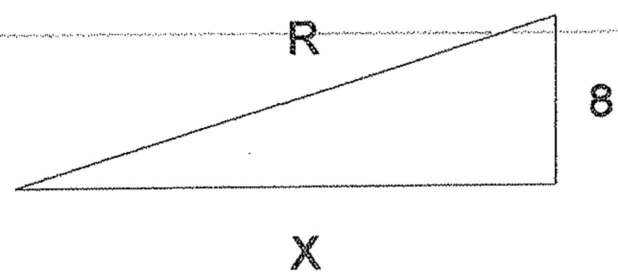
EXAMPLE #3: A dock stands 8 ft. above the deck of a boat. The boat is being pulled into the dock by means of a rope attached to the deck at the front of the boat. If 2 ft of rope is drawn in each minute, at what rate is the boat moving toward the dock when the boat is 15 ft. away?

GIVEN: $x^2 = r^2 - 8^2$
At a given instant: $\frac{dr}{dt} = 2,\ x = 15$

FIND: $\frac{dx}{dt}$

$$\frac{d(x_2)}{dt} = \frac{d(r2-64)}{dt}$$

$$2x\frac{dr}{dt} = 2r\frac{dx}{dt} - 0$$



at x = 15: $15^2 = r^2 - 64$

$$r^2 = 225 + 64$$

$$r^2 = 289$$

$$r = 17$$

$$2(15)\frac{dx}{dt} = 2(17)(2)$$

SUBS: $\frac{dx}{dt} = \frac{34}{15} = 2.27$ ft./min.

RELATED RATES

Sec. 5.7

Example #4: A person 1.8m tall is walking away from a lamppost 5m high at a rate of 2 m/sec. At what rate is the end of the person's shadow moving away from the lamppost?

Given: $\frac{dx}{dt} = 2$ m/sec.

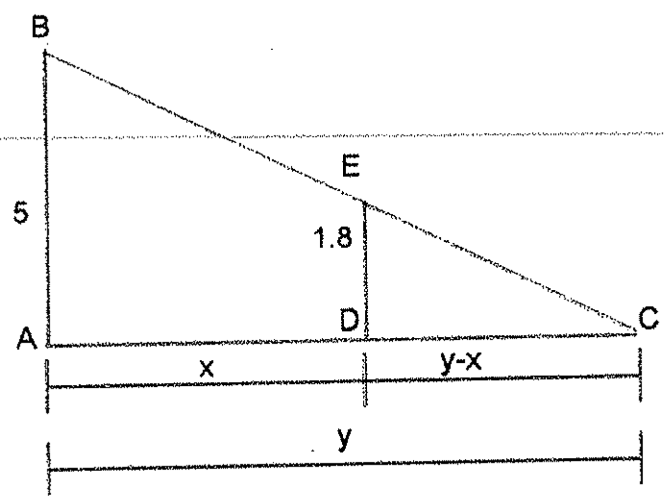
Find: $\frac{dy}{dt}$

From geometry you know that $\triangle ABC$ and $\triangle DEC$ are similar, therefore $\frac{DC}{DE} = \frac{AC}{AB}$

This will give us an equation relating X and Y.

Therefore : $\frac{y-x}{1.8} = \frac{y}{5}$

Solution:



$\frac{y-x}{1.8} = \frac{y}{5}$

$5(y-x)=1.8y$

$5y-5x=1.8y$

$3.2y=5x$

$3.2\frac{dx}{dt} = \frac{dy}{dt}$

$\frac{dy}{dt} = \frac{5}{3.2}\frac{dx}{dt}$

$\frac{dy}{dt} = (\frac{5}{3.2})(\frac{2}{1})$

$\frac{dy}{dt} = \frac{10}{3.2}$

$\frac{dy}{dt} = 3.125$ m/sec.

RELATED RATES

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EXAMPLE #5: When a gas is compressed adiabatically (with no gain or loss of heat) it satisfies the equation $PV^{1.4}=k$, where k is a constant. At a given instant the pressure P is 40 atmospheres and the volume V is 28 liters and is decreasing at a rate of 2 liters/min. At what rate is the pressure changing?

GIVEN: $PV^{1.4}=k$, $P=40$ atm, $V=28l$, $\frac{dv}{dt}=-2l/\text{min}$.

FIND: $\frac{dp}{dt}$

$$PV^{1.4} = k$$

$$P(1.4)V^{.4}\frac{dv}{dt} + V_{1.4}\frac{dp}{dt} = 0$$

$$V^{1.4}\frac{dp}{dt} = -1.4PV_{.4}\frac{dv}{dt}$$

$$(28)^{1.4}\frac{dp}{dt} = (-1.4)(40)(28)^{.4}(-2)$$

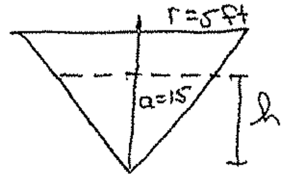
$$106.175\frac{dp}{dt} = 424.7$$

$$\frac{dp}{dt} = \frac{424.7}{106.175}$$

$$\frac{dp}{dt} = 4 \text{ atm/min.}$$

Related Rates

Example #6: Consider a cistern in the shape of an inverted right circular cone where the altitude is 15 ft. and the radius is 5 ft. If water is being pumped in at a rate of 12 ft^3/min, at what rate is the depth h of the water increasing when the depth is 4 ft.



$V = \frac{1}{3}\pi r^2 h$, $\frac{dV}{dt} = 12\,\text{ft}^3/\text{min}$, $h = 4\,\text{ft}$, $\frac{dh}{dt} = ?$

using similar triangles :

$$\frac{\text{radius}}{\text{depth}} = \frac{5}{15} = \frac{1}{3} \Rightarrow r = \frac{1}{3}h$$

$$\therefore\ V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h$$

$$V = \frac{1}{27}\pi h^3$$

$$\frac{dV}{dt} = \frac{1}{9}\pi h^2 \frac{dh}{dt}$$

$$12 = \frac{1}{9}\pi(16)\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{(12)(9)}{(16)(\pi)}$$

$$\frac{dh}{dt} \doteq 2.15\ \text{ft/min}$$

Practice Problem: Same as example #6 except :

alt. = 10 ft, radius = 5 ft, rate = 2 ft/min (rising),
depth = 4 ft.

Question: At what rate is the water entering the cistern?

$V = \frac{1}{3}\pi r^2 h$

$r = \frac{1}{2}h$

$\therefore\ V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h$

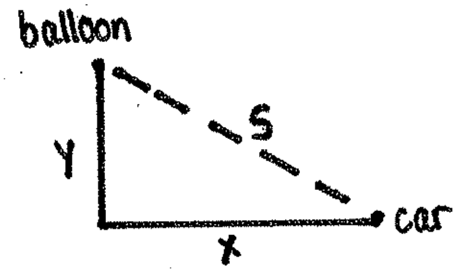
$V = \frac{1}{12}\pi h^3$

$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$

$\frac{dV}{dt} = \frac{1}{4}\pi(16)(2)$

$= 8\pi\ \text{ft}^3/\text{min}$

Example #7: A child riding in a car accidentally releases a helium balloon that rises vertically at 60ft./sec. while the car continues to travel at 80 ft./sec. on the straight road. How fast are the car and balloon separately after 2 sec.; after t seconds.



$s^2 = x^2 + y^2$, $\frac{dx}{dt} = 80$, $\frac{dy}{dt} = 60$, Find $\frac{ds}{dt}$ at $t=2$

at $t=2$, $x=160, y=120 \Rightarrow s^2 = 160^2 + 120^2$

$s=200$

$$s^2 = x^2 + y^2$$
$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$
$$s\frac{ds}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

$$200\frac{ds}{dt} = 160(80) + 120(160)$$
$$\frac{ds}{dt} = 100 \text{ ft./sec.}$$

After t seconds, $x = 80t$, $g = 60t \Rightarrow s = 100t$

$$100t\frac{ds}{dt} = 80t\,(80) + 60t\,(60)$$
$$\frac{ds}{dt} = 100\text{ft./sec.}$$

We conclude that s is increasing at the constant rate of 100ft./sec.

∴ Transcendental Functions

D. The Natural logarithm and its Derivative - Section 6.4

The <u>natural logarithm</u> function is defined by the formula:

$$\boxed{\ln x = \int_1^x \frac{1}{t}\,dt,\ x > 0}$$

Using the chain rule and the Second Fundamental Theorem of Calculus, we find that the general formula is:

$$\frac{d(\ln u)}{dx} = \frac{1}{u} \cdot \frac{du}{dx}$$

Example 1: $y = \ln(x^2+1)$

$u = x^2+1,\ \frac{du}{dx} = 2x$

$$\frac{dy}{dx} = \frac{1}{x^2+1} \cdot 2x$$

$$\boxed{\frac{dy}{dx} = \frac{2x}{x^2+1}}$$

Example 2: $y = \ln \frac{x}{1+x^2}$

$$y = \ln x - \ln(1+x^2)$$

$u = 1+x^2,\ \frac{du}{dx} = 2x$

$$\frac{dy}{dx} = \frac{1}{x} - \frac{2x}{1+x^2}$$

$$\frac{dy}{dx} = \frac{1-x^2}{x(1+x^2)}$$

These rules also work for trig functions.

Example 3: $y = \ln(\cos(4x))$

$u = \cos(4x),\ \frac{du}{dx} = -\sin(4x) \cdot 4$

$$\frac{dy}{dx} = \frac{1}{\cos(4x)} \cdot -4\sin(4x)$$

$$= \frac{-4\sin(4x)}{\cos(4x)}$$

$$\boxed{\frac{dy}{dx} = -4\tan(4x)}$$

Integration formula:

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

Example 4: $\int \frac{3x^2}{x^3+5}\,dx$ $\quad u = x^3+5,\ du = 3x^2\,dx$

$$= \int \frac{1}{u}\,du$$

$$= \ln|x^3+5| + C$$

Example 5: $\int \frac{x^2}{x^3-4}\,dx$ $\quad u = x^3-4,\ du = 3x^2\,dx$

$$= \frac{1}{3}\int \frac{3x^2\,dx}{x^3-4}$$

$$= \frac{1}{3}\int \frac{1}{u}\,du$$

$$= \frac{1}{3}\ln|x^3-4| + C$$

The Exponential Function e^x

$e = 2.718281828459045\ldots$

The rule for the derivative is:

$$\boxed{\frac{d(e^u)}{dx} = e^u \frac{du}{dx}}$$

Example 1: $y = e^{-2x}$

$u = -2x, \ \frac{du}{dx} = -2$

$\frac{dy}{dx} = e^{-2x} \cdot -2$

$$\boxed{\frac{dy}{dx} = -2e^{-2x}}$$

Example 2: $y = e^{x^3}$

$u = x^3; \ \frac{du}{dx} = 3x^2$

$\frac{dy}{dx} = e^{x^3} \cdot 3x^2$

$$\boxed{\frac{dy}{dx} = 3x^2 e^{x^3}}$$

Example 3: $y = e^{\cos x}$

$u = \cos x, \ \frac{du}{dx} = -\sin x$

$\frac{dy}{dx} = e^{\cos x} \cdot -\sin x$

$$\boxed{\frac{dy}{dx} = -\sin x \cdot e^{\cos x}}$$

The rule for the integral is:

$$\boxed{\int e^x \, dx = e^x + C}$$

Example 4: $\int e^{5x} dx$

$u = 5x; \ du = 5dx$

$= \frac{1}{5} \int 5e^{5x} dx$

$$\boxed{= \frac{1}{5} e^{5x} + C}$$

Example 5: $\int e^x(1+e^x)^{1/2}\,dx$

$u=(1+e^x)$, $du=e^x\,dx$

$= \int u^{1/2}\,du$

$= \dfrac{u^{3/2}}{3/2}+C$

$= \frac{2}{3}u^{3/2}+C$

$\boxed{= \frac{2}{3}(1+e^x)^{3/2}+C}$

Example 6: $\int x^3 e^{3x^4}\,dx$

$u=3x^4$, $du=12x^3\,dx$

$= \frac{1}{12}\int 12x^3 e^{3x^4}\,dx$

$\boxed{= \frac{1}{12}e^{3x^4}+C}$

Assignment: AP Calculus p409 (19-26, 43-48)

Integration by Parts

Comes from $y = u \cdot v$

$$d(u \cdot v) = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\Rightarrow u \frac{dv}{dx} = d(u \cdot v) - v \frac{du}{dx}$$

mult by dx

$$\therefore u\, dv = d(u \cdot v)\, dx - v\, du$$

$$\int u\, dv = u \cdot v - \int v\, du$$

Example $\int x \cos(x)\, dx$

$u = x \quad dv = \cos(x)\, dx$

$du = dx \quad v = \sin(x)$

$$\therefore \int x \cos(x)\, dx = x \sin(x) - \int \sin(x)\, dx$$

$$= \boxed{x \sin(x) + \cos(x) + C}$$

- Partial Fractions

The idea is to break the function $\frac{f(x)}{g(x)}$ into smaller fractions to make it easy to integrate.

Example: $\int \frac{1}{x^2+x}\,dx$

we can not do this, therefore we need to change it to something we can work with.

$$\frac{1}{x^2+x} = \frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

mult by the den: $1 = A(x+1) + Bx$

find A+B

$$\text{at } x=0 \Rightarrow 1 = A(1)+0$$
$$A = 1$$
$$\text{at } x=-1 \Rightarrow 1 = 0 + B(-1)$$
$$B = -1$$

$$\therefore \frac{1}{x(x+1)} = \frac{1}{x} + \frac{-1}{x+1}$$
$$= \frac{1}{x} - \frac{1}{x+1}$$

$$\int \frac{1}{x^2+x}\,dx = \int \frac{1}{x}\,dx - \int \frac{1}{x+1}\,dx$$
$$= \ln x - \ln(x+1) + C$$