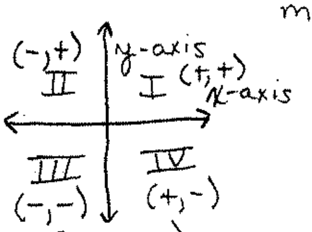
Calculus - Elements of Calculus
by Thomas/Finney

I. Chapter I - The rate of change of a function

A. Coordinates for the Plane - Section 1.1

Cartesian Coordinate System - named in honor of the 17th century math - Decartes



Find Symmetry: P(x,y)

1.) To x-axis - Q(x,-y)
2.) To y-axis - R(-x,y)
3.) To origin - S(-x,-y)
4.) To 45° line - T(y,x)

Find Q,R,S,+T for the point P(1,-2)

Q (1,2)
R (-1,-2)
S (-1,2)
T(-2,1)

Assignment: Calculus - p4 (2,4,6,8,16,17)
AP Calculus - p4(4,9,16,19)

B. The Slope of a Line - Section 1.2

$$\text{Slope} = m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_f - y_I}{x_f - x_I}$$

Give examples

Angle of Inclination - the smallest angle you get measure counter clockwise from the x-axis

The slope of a line is the tangent of the line's angle of inclination.

Cases:

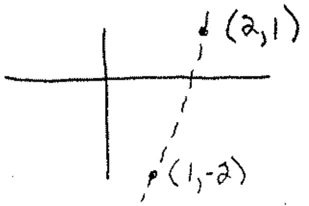
1.) $m = \frac{\text{rise}}{\text{run}} = \frac{y_2-y_1}{x_2-x_1} = \frac{\Delta y}{\Delta x}$
2.) $m = \tan\theta$ (θ = the angle of inclination)
3.) Vertical lines have no slope
4.) Horizontal lines have 0 slope
5.) Parallel lines: $m_1 = m_2$
6.) Perpendicular lines: $m_2 = \frac{-1}{m_1}$ or $m_1 m_2 = -1$

Find the slope of points A+B, and then find the slope of a line $\perp$ to $\overline{AB}$.

$A(1,-2)$, $B(2,1)$

$m = \frac{1--2}{2-1} = \frac{3}{1} = 3$

$m_\perp = -\frac{1}{3}$



Assignment: Calculus - p 9-10 (4, 13, 14, 19, 21, 23, 27, 36, 39)
AP Calculus - p 9-10 (4, 13, 14, 19, 39, 43) look at 36+37

C. Equations for lines - Section 1.3

Equations Needed:

1.) Vertical line through (a,b): $x = a$
2.) horizontal line through (a,b): $y = b$
3.) Slope-intercept: $y = mx + b$
4.) Point-slope: $y - y_1 = m(x - x_1)$
5.) General linear equation: $Ax + By = C$
6.) Point-to-Point distance formula: $d = \sqrt{\Delta x^2 + \Delta y^2}$
7.) Point-to-Line distance formula:

$$d = \frac{|Ax + By + C|}{\sqrt{A^2 + B^2}}$$

8.) Midpoint of a line through $(x_1,y_1)+(x_2,y_2) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

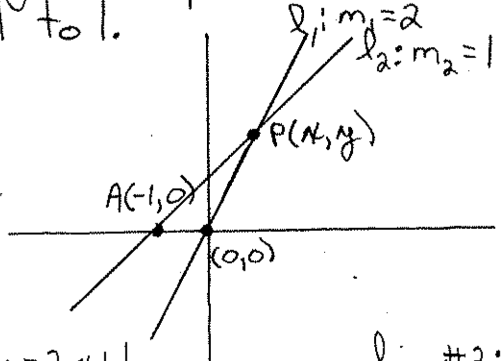
Examples: Given 2 points $(2,5)$ + $(-2,-10)$, Find the equation of the line.

Solution: 1.) Find the slope, 2) use one point + slope to find the equation.

1.) $m = \frac{y_2-y_1}{x_2-x_1} = \frac{-10-5}{-2-2} = \frac{-15}{-4} = \frac{15}{4}$

2.) $y - 5 = \frac{15}{4}(x-2)$
$4y - 20 = 15x - 30$
$15x - 4y = 10$

Example: Find the coordinates of a point $P(x,y)$ which is so located that the line l_1 through the origin and P has a slope equal to 2, and the line l_2 through the point $A(-1,0)$ and P has a slope equal to 1.



Key: $y=mx+b$

line #1: $y=2x+b$
$0=2(0)+b$
$b=0$
$y=2x$

line #2: $y=1x+b$
$0=1(-1)+b$
$b=1$
$y=x+1$

$\therefore\ 2x=x+1$
$x=1$
$y=2x=2(1)=2$

$\boxed{P(1,2)}$

Example: Find the distant from the points $h(-2,0)$, $j(1,1)$, $k(5,8)$ and the line $3x+5y-8=0$

$$h:\ d=\frac{|3(-2)+5(0)-8|}{\sqrt{9+25}}=\frac{|-6-8|}{\sqrt{34}}=\frac{|-14|}{\sqrt{34}}=\frac{14}{\sqrt{34}}\quad (2.4010)$$

$$j:\ d=\frac{|3(1)+5(1)-8|}{\sqrt{34}}=\frac{|0|}{\sqrt{34}}=0$$

$$k:\ d=\frac{|3(5)+5(8)-8|}{\sqrt{34}}=\frac{|15+40-8|}{\sqrt{34}}=\frac{|47|}{\sqrt{34}}=\frac{47}{\sqrt{34}}\quad (8.0605)$$

when you substitute into the equation:
$h_n \Rightarrow -14$, this indicates h is below l
$j_n \Rightarrow 0$, this indicates j is on l
$k_n \Rightarrow 47$, this indicates k is above l
(graph these on the board)

Assignment: Calculus - p16-17 (1, 9, 11, 15, 24, 29, 35, 39, 43, 53, 56, 80)
AP Calculus - p16-17 (~~8~~, 24, 25, 41, 44, 57, 60, ~~76~~, ~~81~~, 82)
9

D. Functions and Graphs - Section 1.4

In $y = f(x)$:

<u>Domain</u> is the values the x can have
<u>Range</u> is the values the y can have

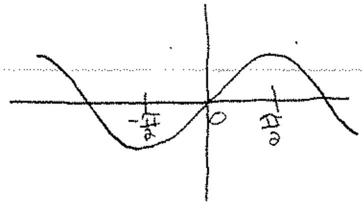
4 types of Domain + Range
1.) Open interval : $1 < x < 2$ or $(1, 2)$
2.) Closed interval : $1 \le x \le 2$ or $[1, 2]$
3.) half-open interval: $1 < x \le 2$ or $(1, 2]$
4.) Infinite interval : $-\infty < x \le 2$ or $(-\infty, 2]$

In $y = f(x)$, x is the independent variable and
y is the dependent variable

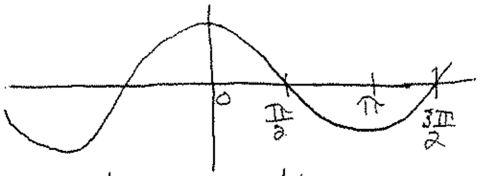
<u>Steps in Graphing</u>
1.) Make a table of values
   a.) Make sure you pick some points where the function crosses or touches the $x + y$ axis.
   b.) Also, pick some points at or near any endpoints in the domain.
2.) Plot the points on a graph
3.) Sketch the graph by connecting the points

<u>Trig functions</u>

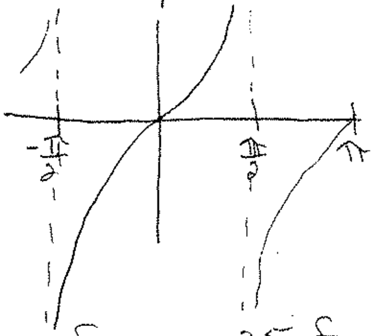
$y = \sin x$



$y = \cos x$



$y = \tan x = \frac{\sin x}{\cos x}$



(see table on top of page 25 for values)

Sums, Differences, Products, and Quotients of Functions

The domain of Sum, difference, and product is found by taking the intersection of the domains of the 2 functions. For the Quotient, you must also make restriction when the new function is undefined.

Composition of functions

Given: $f(x) = x^3$ and $g(x) = x + 2$

Find: $g(f(x))$ and the value for $g(f(3))$

$$g(x) = x+2$$
$$g(f(x)) = f(x) + 2$$
$$= x^3 + 2$$
$$g(f(3)) = 3^3 + 2 = 29$$

Assignment: Calculus p28-29 (3, 6, 13, 18, 25, 34, 47, 50)
AP Calculus p28-29 (20, 29, 34, 35, 47, 50)

E. Absolute Values - Section 1.5

Absolute value of $x = |x| = \sqrt{x^2}$ (always positive)

<u>Rules for absolute values:</u>

1.) $|x| = \sqrt{x^2}$
2) $|xy| = |x| \cdot |y|$
3.) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$
4.) $|x-y| = |y-x|$
5.) $|x+y| \leq |x| + |y|$

Intervals

6.) $|x| < a \Leftrightarrow -a < x < a$
7.) $|x-b| < a \Leftrightarrow b-a < x < b+a$

Find the intervals:

Example: $|x| < 2$
$$-2 < x < 2$$

Example: $|3x-6| < 1$
$$-1 < 3x-6 < 1$$
$$5 < 3x < 7$$

$\frac{5}{3} < x < \frac{7}{3}$

Solve: $|x| + |x-1| = 3$

$\sqrt{x^2} + \sqrt{(x-1)^2} = 3$

$\sqrt{x^2} = 3 - \sqrt{(x-1)^2}$

$x^2 = 9 - 6\sqrt{(x-1)^2} + (x^2-2x+1)$

$x^2 = x^2 - 2x + 10 - 6\sqrt{(x-1)^2}$

$6\sqrt{(x-1)^2} = 10 - 2x$

$36(x-1)^2 = 100 - 40x + 4x^2$

$36x^2 - 72x + 36 = 100 - 40x + 4x^2$

$32x^2 - 32x - 64 = 0$

$32(x^2 - x - 2) = 0$

$32(x-2)(x+1) = 0$

$\Rightarrow x = 2$ or -1 (you must check these in the original)

Note: $(g \circ f)(x)$ means $g(f(x))$

Assignment: Calculus p34-35 (6, 7, 29, 31, 34, 46, 47, 51)
AP Calculus p34-35 (6, 7, 29, 31, 34, 46, 47, 51)

F. Tangent Lines - Section 1.6

The tangent line is a rate of change - this is what diff calculus is all about. We are interested in find the slope of the tangent line.

For a straight line, finding the slope is no problem.

$y = 2x + 4 \Rightarrow$ slope $= 2$

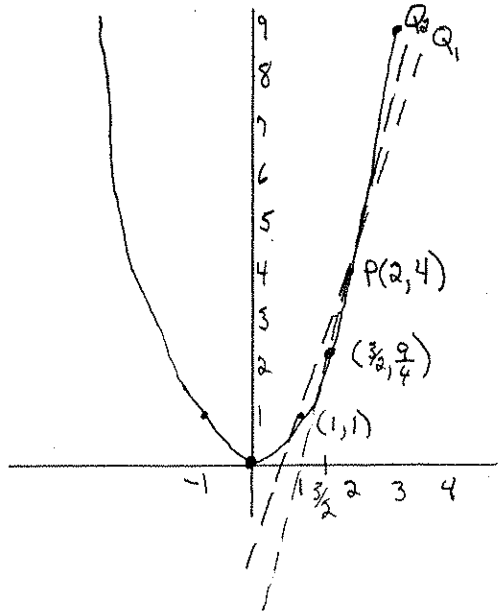
Finding the slope for a curved line is another story. The slope of the line tangent to the curve changes as you move along the curve.

Fermat found that you could use the slope of the secant line to get an idea about the slope of the tangent line.

Example: Find the slope of the line tangent to the curve $y = x^2$ at the point $P(2,4)$. Find the eq. for tangent line

(NOTE: The slope of the Secant line = Average Rate of Change, and The slope of the tangent line = Instanous Rate of Change)

x	y
0	0
1	1
2	4
3	9
−1	1
$\frac{3}{2}$	$\frac{9}{4}$



$$m_{Q_1} = \frac{\Delta y}{\Delta x} = \frac{4-1}{2-1} = 3$$

$$m_{Q_2} = \frac{\Delta y}{\Delta x} = \frac{4-\frac{9}{4}}{2-\frac{3}{2}} = \frac{\frac{7}{4}}{\frac{1}{2}} = \frac{7}{2} = 3.5$$

If you continue this, you will get very close to the slope of the tangent line.

We can do this mathematically using the slope of the secant line : $m_{sec} = \frac{\Delta y}{\Delta x} = \frac{y_F - y_I}{x_F - x_I}$

$$m_{sec} = \frac{\Delta y}{\Delta x}$$

$$\Delta y = (x + \Delta x)^2 - x^2$$

$$\Delta x = \Delta x = \text{change in } x$$

$$\therefore \quad m_{sec} = \frac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x}$$

$$m_{sec} = \frac{2x\Delta x + \Delta x^2}{\Delta x} = 2x + \Delta x$$

As Δx gets very small, we find the equation for the slope of the tangent line : $m_{tan} = 2x$

$\therefore$ at $P(2,4)$, the slope of the tangent line $= 2x = 4$

Find the equation of the tangent line :

$$y - y_1 = m(x - x_1)$$

$$y - 4 = 4(x - 2)$$

$$y - 4 = 4x - 8$$

~~$y = 2x$~~

$$y = 4x - 4$$

Find the equation of the tangent line at $P(-1,1)$

$$m_{tan} = 2x = 2(-1) = -2$$

$$y - y_1 = m(x - x_1)$$
$$y - 1 = -2(x - (-1))$$
$$y - 1 = -2(x+1)$$
$$y - 1 = -2x - 2$$
$$y = -2x - 1$$

(Does this make sense?)

Assignment: Calculus p 41-42 (2, 3, 5, 8, 13, 14, 15, 19)
AP Calculus p 41-42 (2, 3, 7, 12, 16, 19)

G. Slope of the curve $y = f(x)$: Derivatives - Section 1.7

This is known as differential calculus.
A derivative is the rate of change of a function at any given point (or instant). Instantanous rate of change.
Symbols for derivatives: $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, y', $f'(x)$, $\frac{dy}{dx}$

4 step process (Delta Process) for finding the derivative, given: $y = f(x)$

1.) Determine $f(x + \Delta x)$
2.) Calculate $f(x+\Delta x) - f(x)$
3.) Divide by Δx
4.) Let $\Delta x \to 0$ and evaluate

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = m_{tan}$$

Examples: Find $f'(x)$ of the functions, Find the slope of the tangent line at $x=2$, and Find the equation of the tangent line at $x=2$

Note: You may need to remember
$(a-b)^2 = a^2 - 2ab + b^2$
$(a+b) = a^2 + 2ab + b^2$
$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$
$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

1.) $f(x) = x^2 + \frac{1}{x}$

$$f(x+\Delta x) = (x+\Delta x)^2 + \frac{1}{x+\Delta x} = x^2+2x\Delta x+\Delta x^2 + \frac{1}{x+\Delta x}$$

$$f(x+\Delta x) - f(x) = x^2+2x\Delta x+\Delta x^2 + \frac{1}{x+\Delta x} - x^2 - \frac{1}{x}$$

$$= 2x\Delta x + \Delta x^2 + \frac{x}{x(x+\Delta x)} - \frac{x+\Delta x}{x(x+\Delta x)}$$

$$= 2x\Delta x + \Delta x^2 - \frac{\Delta x}{x(x+\Delta x)}$$

$$= \Delta x\left(2x + \Delta x - \frac{1}{x(x+\Delta x)}\right)$$

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = 2x + \Delta x - \frac{1}{x(x+\Delta x)}$$

$$\lim_{\Delta x \to 0}\left(2x+\Delta x - \frac{1}{x(x+\Delta x)}\right) = 2x - \frac{1}{x^2}$$

$$f'(x) = 2x - \frac{1}{x^2} = m_{tan}$$

at $x=2$, $m_{tan} = 2(2) - \frac{1}{4} = 3.75$ or $3\frac{3}{4}$ or $\frac{15}{4}$

$y = f(x) = x^2 + \frac{1}{x}$; at $x=2$, $y = (2)^2 + \frac{1}{2} = 4.5$

$\therefore$ we have the point $P(2, 4.5)$

$$y - y_1 = m(x - x_1)$$
$$y - 4.5 = 3.75(x-2)$$
$$y - 4.5 = 3.75x - 7.5$$
$$y = 3.75x - 3$$

2.) $f(x) = \sqrt{x+1}$

$$f(x+\Delta x) = \sqrt{x+\Delta x+1}$$

$$f(x+\Delta x) - f(x) = \sqrt{x+\Delta x+1} - \sqrt{x+1}$$

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{\sqrt{x+\Delta x+1} - \sqrt{x+1}}{\Delta x}$$

(Note: the Key here is to rationalize the numerator)

$$= \frac{(\sqrt{x+\Delta x+1} - \sqrt{x+1})}{\Delta x} \cdot \left(\frac{\sqrt{x+\Delta x+1} + \sqrt{x+1}}{\sqrt{x+\Delta x+1} + \sqrt{x+1}}\right)$$

$$= \frac{x+\Delta x+1-x-1}{\Delta x(\sqrt{x+\Delta x+1} + \sqrt{N+1})}$$

$$= \frac{\Delta x}{\Delta x(\sqrt{x+\Delta x+1} + \sqrt{x+1})}$$

$$= \frac{1}{\sqrt{x+\Delta x+1} + \sqrt{x+1}}$$

$$\lim_{\Delta x \to 0} \left(\frac{1}{\sqrt{x+\Delta x+1} + \sqrt{x+1}} \right) = \frac{1}{2\sqrt{x+1}}$$

$$f'(x) = \frac{1}{2\sqrt{x+1}} = m_{tan}$$

at $x=2$, $m_{tan} = \frac{1}{2\sqrt{2+1}} = \frac{1}{2\sqrt{3}} \doteq \frac{1}{3.4641} \doteq .288675$

$y = f(x) = \sqrt{x+1}$; at $x=2$, $y = \sqrt{3} \doteq 1.732$

P(2, 1.732)

$$y - y_1 = m(x - x_1)$$
$$y - 1.732 = .288675(x-2)$$
$$y - 1.732 = .288675x - .57735$$
$$y = .288675x + 1.15465$$

Assignment: Calculus p48 (2, 4, 5, 17, 19)
AP Calculus p48 (4, 7, 17)

H. Velocity and Other Rates of Change - Section 1.8

Average rate of change = $\frac{f(x+\Delta x) - f(x)}{\Delta x}$

Instantaneous rate of change = $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

Equation for Free Fall: $S = S_0 + V_0 t + \frac{1}{2} a t^2$

Systems of measure (a is near the earth's surface)

English: S = ft, t = sec., V = ft/sec, a = 32 ft/sec^2

MKS: S = meters, t = sec, V = m/sec, a = 9.8 m/sec^2

CGS: S = cm, t = sec, V = cm/sec, a = 980 cm/sec^2

Application Example:

A rock is thrown upward from the ground. The rock's distance is given by the equation: $S = 144t - 16t^2$

a.) Find the rock's average velocity at $t = 2$ sec
b.) Find the rock's instantaneous velocity at $t = 2$sec
c.) When will the rock reach its max. altitude?
d.) How high will it be?
e.) When will the ball hit the ground?

a.) $V_{ave} = \frac{\Delta S}{\Delta t}$ $\quad t = 0 \Rightarrow S = 144(0) - 16(0)^2 = 0$ ft.
$t = 2 \Rightarrow S = 144(2) - 16(4) = 224$ ft
$\Delta t = 2 - 0 = 2$

$$\boxed{V_{ave} = \frac{224 - 0}{2 - 0} = 112 \text{ ft/sec}}$$

b.) $V_{ins} = \lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t} = 144 - 32t$

$$\boxed{V_{ins} = 144 - 32(2) = 80 \text{ ft/sec}}$$

c.) max. alt. when $m_{tan} = 0 \Rightarrow V_{ins} = 0$

$\therefore$ $0 = 144 - 32t$
$$\boxed{t = 4.5 \text{ sec}}$$

d.) $S = 144(4.5) - 16(4.5)^2$
$S = 648 - 324$
$$\boxed{S = 324 \text{ ft.}}$$

e.) The rock will hit the ground when $S = 0$, therefore

$0 = 144t - 16t^2$
$16t(9 - t) = 0$
$$\boxed{t = 0 \text{ or } t = 9}$$

Assignment: Calculus p53-55 (3, 6, 8, 21)
AP Calculus p53-55 (2, 5, 10, 11, 20, 21)

(Note: Look at problems 15-19 on pps. 54-55)

Pages 54-55, Problems 15-19 (All from Graph)

15.) From the graph, we see that at $t=2$ sec, the rocket had a velocity of 190 ft/sec

16.) The engine burned for 2 sec

17.) Highest point reached at $t=8$ sec, inst vel $=0$

18.) at $t=10.8$ sec., falling at a rate of 90 ft/sec down

19.) 2.8 sec.

The equation(s) for the distance the rocket travels, S in terms of t, must be divided into 3 steps:

1.) for $t=0$ to $t=2$ $[0,2]$
2.) for $t>2$ to $t<10.8$ $(2,10.8)$
3.) for $t=10.8$ to $t=13$ $[10.8,13]$

At this time we can not find S in terms of t for Step #1 because, $F=ma$ yields a variable acc while the rocket is under power. The force due to acc of the engine is constant while the mass of the rocket is decreasing. This gives us an increasing acc.

I. Limits - Section 1.9

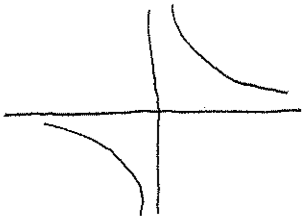
Limits: The value a function reaches when its variable approaches a given value.

* NOTE: The limit only exists if it approaches the same value from both sides.

Right-hand limit - approaches a value from the positive side
$\lim_{x\to c^+} f(x)$

Left-hand limit - approaches a value from the negative side
$\lim_{x\to c^-} f(x)$

Two-sided limit - $\lim_{x\to c} f(x)$



Example: $\lim_{x\to 0} \frac{1}{x}$ (does not exist)

$\lim_{x\to 0^+} \frac{1}{x} = \infty$ $\lim_{x\to 0^-} \frac{1}{x} = -\infty$

Properties of limits:

1.) $\lim_{x\to c} b = b$ 2.) $\lim_{x\to c} x = c$

3.) $\lim_{x\to c} b\cdot f(x) = b\cdot \lim_{x\to c} f(x)$ 4.) $\lim_{x\to c} [f(x) \pm g(x)] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x$

5.) $\lim_{x\to c} [f(x)\cdot g(x)] = [\lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)]$

6.) $\lim_{x\to c} x^n = c^n$

7.) $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, provided $\lim_{x\to c} g(x) \neq 0$

Examples:

1.) $\lim_{x\to 0} (7x-4) = (7(0)-4) = -4$

2.) $\lim_{x\to 1} \frac{2x^2-5x+3}{x-1} = \lim_{x\to 1} \frac{(2x-3)(x-1)}{(x-1)} = \lim_{x\to 1} (2x-3)$

$= (2(1)-3) = -1$

Chapter 1 Sections 9 and 10

Limits are one of the easiest Calculus problems to do (at least to this point)! You have only 2 types. They are:

Chapter 1 Section 9

1.) The limit of a function as its variable goes to a specific number. In this case, you have 3 steps to follow.

a. Put the specific number into the expression to see if you get an answer. If you get a specific answer such as 4, then you are done and you have the answer. If you get something that is undefined, positive infinity, or negative infinity; you must try step b.

b. At this point, you must try to factor the expression (both numerator and denominator). If you are lucky, it will factor and you can then substitute the value into the new expression and get your answer. If it does not factor easily, you must try long division on the problem to find the factors. Once this is done, you can substitute the vlue into the expression and get your answer.

c. If both of these fail, you must try the right and left handed limit to see if they converge at a given value. If they converge at the same value, then that is the limit. If not, then the limit Does Not Exist (DNE)!

Chapter 1 Section 10

2.) The limit of a function as its variable goes to infinity. In this case you will simply divide the numerator and the denominator by the variable with the highest exponent in the denominator. Once you have done this correctly, use the definition that Lim(1/X) = 0 as X goes to infinity. This should give you the answer.

Squeeze Theorem

If $3x \le f(x) \le x^3 + 2$ for $0 \le x \le 2$, evaluate $\lim_{x \to 1} f(x)$

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) =$$

$$\lim_{x \to 0} x^2 \cdot \cos\left(\frac{1}{x^2}\right) =$$

$$-1 \le \cos\left(\frac{1}{x^2}\right) \le 1$$

$$\Rightarrow -x^2 \le x^2 \cos\left(\frac{1}{x^2}\right) \le x^2$$

$$\lim_{x \to 0} -x^2 \le \lim_{x \to 0} x^2 \cos\left(\frac{1}{x^2}\right) \le \lim_{x \to 0} x^2$$

$$0 \le \lim_{x \to 0} x^2 \cos\left(\frac{1}{x^2}\right) \le 0$$

$$\boxed{\therefore \lim_{x \to 0} x^2 \cos\left(\frac{1}{x^2}\right) = 0}$$

Trig limits: Note: θ is measured in radians

$$\lim_{\theta \to 0} \sin\theta = 0$$

$$\lim_{\theta \to 0} \cos\theta = 1$$

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

$$\lim_{\theta \to 0} \frac{3\tan\theta}{\theta} = 3\lim_{\theta \to 0} \frac{\tan\theta}{\theta} = 3\lim_{\theta \to 0} \left(\frac{\sin\theta}{\theta}\right)\left(\frac{1}{\cos\theta}\right)$$

$$= 3 \cdot 1 \cdot \frac{1}{1} = 3$$

Assignment: Calculus p 68-69 (1-10, 31-36)
AP Calculus p 68-69 (6, 10, 20, 29, 35, 36, 38, 41, 49-51)
Note: look at page 70 (71-86)

J. Infinity as a Limit - Section 1.10

In this section you use all the tricks you learned in section 1.9.
One additional trick is dividing the num. and den. by the same thing. Also, $\lim_{x \to \infty} \frac{1}{x} = 0$

Example: $\lim_{x \to +\infty} \frac{3-2x^4}{x+1}$ (divide both by: x^4)

This will give you: $\lim_{x \to +\infty} \frac{3/x^4 - 2}{1/x^3 + 1/x^4}$

num: $\lim_{x \to +\infty} 3/x^4 - 2 = 0 - 2 = -2$

den: $\lim_{x \to +\infty} 1/x^3 + \frac{1}{x^4} = 0 + 0 = 0$

$\lim_{x \to +\infty} \frac{-2}{0} = -\infty$

$$\boxed{\frac{m}{m_0} = \frac{1}{\sqrt{1 - \frac{V^2}{V_c^2}}}}$$

Example: $\lim_{x \to \infty} \frac{4x^2 - x}{2x^3 - 5}$

$= \lim_{x \to \infty} \frac{4/x - 1/x^2}{2 - 5/x^3}$

$= \frac{0-0}{2-0} = \frac{0}{2} = 0$

Assignment: Calculus page 76 (1, 4, 7-9, 14-16, 23-26)
APCalculus p 76-77 (2, 5, 7-9, 13-15, 20-22, 27, 31, 43, 44, 49-51)

K. Continuity - Section 1.11

Intro: A moving physical object cannot vanish at some point and reappear someplace else to continue its motion. Thus, we perceive the path of a moving object as a single, unbroken curve without gaps, jumps, or holes. Such curves can be described as "continuous".

The Continuity Test:

The function $y = f(x)$ is continuous at $x = c$ if and only if all 3 conditions are true:

1.) $f(c)$ exists
2.) $\lim_{x \to c} f(x)$ exists
3.) $\lim_{x \to c} f(x) = f(c)$

If one of these conditions fail, the function is said to be discontinuous at c, and c is called a point of discontinuity.

If condition 3 holds, then 1 and 2 hold automatically.

Note: Polynomials are continuous functions.
A rational function is continuous everywhere except at the points where the denominator is zero.
A function is continuous at every point at which it has a derivative.

Exercises (1,2) (Worksheet)

1.) a.)

x	$f(x)=x^2+3$
1.) 2	7
2.) 2.5	9.25
3.) 2.7	10.29
4.) 2.8	10.84
5.) 2.9	11.41
6.) 2.99	11.9401
7.) 2.999	11.994

x	$f(x)=x^2+3$
1) 4	19
2) 3.6	15.96
3) 3.3	13.89
4) 3.1	12.61
5) 3.01	12.0601
6) 3.001	12.006
7.) 3.0001	12.0006

b.) $\lim_{x\to 3}(x^2+3) = 12$

2.) a.) $\lim_{x\to 2}\frac{x^2-5x+6}{x-2} = \lim_{x\to 2}\frac{\cancel{(x-2)}(x-3)}{\cancel{(x-2)}} = \lim_{x\to 2}(x-3) = -1$

b.) $\lim_{x\to 1}\frac{x^2-1}{x^3-1} = \lim_{x\to 1}\frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}(x^2+x+1)} = \lim_{x\to 1}\frac{x+1}{x^2+x+1} = \frac{2}{3}$

c.) $\lim_{x\to 4}\frac{2-\sqrt{x}}{4-x} = \lim_{x\to 4}\frac{\cancel{(2-\sqrt{x})}}{\cancel{(2-\sqrt{x})}(2+\sqrt{x})} = \lim_{x\to 4}\frac{1}{2+\sqrt{x}} = \frac{1}{2+2} = \frac{1}{4}$

d.) $\lim_{x\to\infty}\frac{(2x-4)}{5x} = \lim_{x\to\infty}\frac{2-\frac{4}{x}}{5} = \frac{2}{5}$

e.) $\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1} = \lim_{x\to 1}\frac{(\sqrt{x}+1)\cancel{(\sqrt{x}-1)}}{\cancel{(\sqrt{x}-1)}} = \lim_{x\to 1}(\sqrt{x}+1) = 1+1 = 2$

f.) $\lim_{x\to -3} f(x)$, where $f(x) = \begin{cases} 2x-1, & \text{if } x > -3 \\ x+3, & \text{if } x < -3 \\ 5, & \text{if } x = -3 \end{cases}$

$\lim_{x\to -3^-} f(x) = \lim_{x\to 3^-}(x+3) = 0$

$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+}(2x-1) = -7$

$\therefore \lim_{x\to -3} f(x) =$ does not exist

18.) $f(x) = \begin{cases} -2x, & x \le 2 \\ x^2-4x+1, & x > 2 \end{cases}$

Claim: discon at $x=2$

1.) $f(c)$ is defined

$f(2) = -2(2) = -4$

2.) $\lim_{x\to 2} f(x) =$ exists

$$\left.\begin{aligned} &\lim_{x\to 2^+}(x^2-4x+1) = 4-8+1 = -3 \\ &\lim_{x\to 2^-}(-2x) = (-2)(2) = -4 \end{aligned}\right\} \therefore \lim_{x\to 2} f(x) = \text{does not exist}$$

∴ discon at $x=2$, Non-removable, jump or break in graph

31.) $f(x) = \frac{x^2-16}{x-4}$

Claim: discon at $x=4$

1.) $f(c)$ is defined

$f(4) = \frac{16-16}{4-4} = \frac{0}{0}$?

2.) $\lim_{x\to 4} f(x) =$ exists

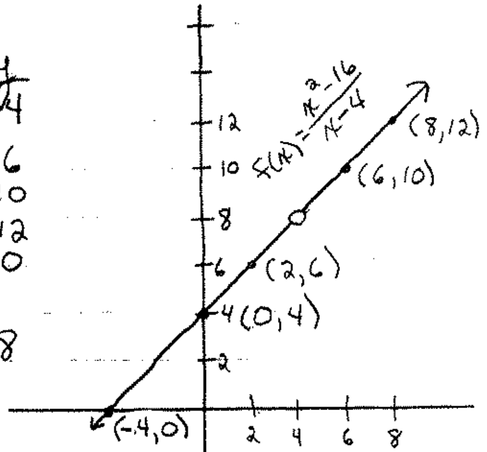
$$\lim_{x\to 4}\frac{x^2-16}{x-4} = \lim_{x\to 4}\frac{(x+4)\cancel{(x-4)}}{\cancel{(x-4)}} = \lim_{x\to 4}(x+4) = 8$$

3.) $\lim_{x\to c} f(x) = f(c)$

$8 = \frac{0}{0}$

∴ discon at $x=4$, removable, hole in graph at $x=4$.

x	y
0	4
2	6
6	10
8	12
-4	0



(from graph, discon at $x=4$)

36.) From the graph, $f(x) = x\sqrt{x+3}$ is continuous for $\{x \mid x \ge -3\}$ $[-3,\infty)$

Ch1.11 - Continuity

A moving object cannot vanish at some point and reappear someplace else

To be continuous, the graph must contain no breaks, no jumps, no holes, no asymptotes.

Def: A function is said to be continuous at some point $x = c$ if the following 3 conditions are met:

1.) $f(c)$ is defined

2.) $\lim_{x \to c} f(x)$ exists

3.) $\lim_{x \to c} f(x) = f(c)$

Theorems

1.) All polynomial functions are continuous at every pt.

2.) Rational functions are discontinuous at every pt not in the domain of the function.

Types of discontinous functions

1.) Removable discon - if $f(x)$ can be made continous by defining $f(x)$ at $x = c$

(hole in graph) (occurs when $f(c) = \frac{0}{0}$)

2.) Non-removable - if $f(x)$ cannot be defined.

$\lim_{x \to c} f(x) = \frac{N}{0}$ (asymptote)

$\lim_{x \to c} f(x) = \text{DNE}$ (Jump or Break)

Examples:

1) Find all points of discon for $f(x) = \frac{x^3 - 1}{x - 1}$

2.) Determine if $f(x) = x^2$ is continuous at $x = 1$

3) Find any discon for $f(x) = \frac{x-1}{x^2 - x}$

4) Find any discon for $\begin{cases} -2x, & x \leq 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$

Intermediate Value Theorem: (IVT) If $f(x)$ is continuous on $[a, b]$, and k is some number between $f(a)$ and $f(b)$, then there is at least one number c, between a and b, such that $f(c) = k$

Ex: Use "IVT" to verify that $f(x) = x^3 + x - 3$ has a real zero between $[1, 2]$

Determine the points at which the functions are discontinuous and state the type of discontinuity

1.) $f(x) = \frac{1}{x}$

2.) $f(x) = \frac{x-2}{|x-1|}$

3.) $f(x) = \frac{1}{x^2-1}$

4.) $f(x) = \frac{1-2x}{x^2-x-6}$

5.) $f(x) = \frac{x+1}{4x-2}$

Max-Min Theorem (This is important for later)
If f is continuous at every point on the closed inteval [a, b], the f takes on a max. and min. value over the closed inteval [a, b]

Example 1: Show that $f(x) = x^2 - 2x + 1$ is a continuous function.

$$\lim_{x \to c} f(x) = \lim_{x \to c} (x^2 - 2x + 1) = c^2 - 2c + 1 = f(c)$$

Example 2: Find the points of discontinuity

$$f(x) = \frac{x-4}{x^2-16}$$

The easiest way is to find if the den. is ever 0.

$$\therefore x^2 - 16 = 0$$
$$x = \pm 4$$

Assignment: Calculus p86-87 (1-6, 16-19)
APCalculus p86-87 (1-6, 14-17, 21-23)

L. Chapter 1 Test
Look at pages 88-91 Review Questions and Miscellaneous Problems.

II. Chapter 2 - Derivatives

A. Polynomial Functions and Their Derivatives - Section 2.1

Items you need to remember:

1.) The equation for the slope of a line tangent to a curve at any given point is the derivative of the curve.
2.) The derivative of distance with respect to time is the instantanous velocity.
3.) The derivative of velocity with respect to time (or $\frac{d^2s}{dt^2}$) is the equation for acc.
4.) Two lines are parallel if $m_1 = m_2$
5.) Two lines are perpendicular if $m_1 \cdot m_2 = -1$
6.) The units for distances, time, vel., and acc. in the English, MKS, and CGS systems.

<u>Rules for differentiation</u> - learn these, there will be more to come.

1.) $\frac{d(C)}{dx} = 0$, where C is any constant

2.) (Simple) Power Rule

$$\frac{d(x^n)}{dx} = nx^{n-1}$$

3.) (Constant) Multiple Rule

$$\frac{d(cf(x))}{dx} = cf'(x)$$

4.) Sum Rule

$$\frac{d(f(x)+g(x))}{dx} = f'(x) + g'(x)$$

(The derivative of a sum is equal to the sum of the derivatives)

Do a few examples using the Delta Process and the rules.

1.) $y = f(x) = x + 3$ 2.) $y = f(x) = x^2 + 4$

Second Derivative = the derivative of the first derivative.
This can be written the following ways:

1.) y'' (where $y' = \frac{dy}{dx}$)
2.) $\frac{d^2y}{dx^2}$
3.) $\frac{d}{dx}\left(\frac{dy}{dx}\right)$

Example 1: Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ of $y = (3x+1)^3$

$$y = (3x)^3 + 3(3x)^2(1) + 3(3x)(1)^2 + 1^3$$
$$y = 27x^3 + 27x^2 + 9x + 1$$
$$\frac{dy}{dx} = 81x^2 + 54x + 9$$
$$\frac{d^2y}{dx^2} = 162x + 54$$

Example 2: Find the tangent to the curve $y = x^2 + 5x$ at $(3, 24)$ and then find the line $\perp$ to the tangent at $(3, 24)$.

$$y = x^2 + 5x$$
$$\frac{dy}{dx} = 2x + 5 = m_1$$
at $x = 3$, $m_1 = 2(3) + 5 = 11$
$y - y_1 = m(x - x_1)$ – point slope
$$y - 24 = 11(x - 3)$$
$$y - 24 = 11x - 33$$
$\boxed{y = 11x - 9}$ tangent

if $m_1 = 11$, then $m_2 = -\frac{1}{11}$
$$y - 24 = -\frac{1}{11}(x - 3)$$
$$y - 24 = -\frac{1}{11}x + \frac{3}{11}$$
$\boxed{y = -\frac{1}{11}x + \frac{267}{11}}$ $\perp$ to tangent

Assignment: Calculus p99-100 (12, 15, 16, 20, 26, 29, 33, 38)
AP Calculus p99-100 (13, 14, 19, 22, 26, 29, 32-36, 38, 41)

B. Products, Powers, and Quotients - Section 2.2

1.) General Power Rule

$$\frac{d(u^n)}{dx} = n u^{n-1} \frac{du}{dx}$$

Example: $y = (x^2)^3$ find $\frac{dy}{dx}$

There are 2 ways of doing this simple problem

a.) $y = x^6$

$\frac{dy}{dx} = 6x^5$

b.) $y = (x^2)^3$, $u = x^2$, $n = 3$

$\Rightarrow \frac{dy}{dx} = (3)(x^2)^2(2x)$

$= 6x^5$

In words: exponent X base to the 1 less power X der. of base

Example: $y = (x^2+4x-1)^3$

$\frac{dy}{dx} = (3)(x^2+4x-1)^2(2x+4)$

$= 6(x+2)(x^2+4x-1)^2$

Note: This also works for neg. exponents

2.) Product Rule : first X der. of second + second X der of first

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + V\frac{du}{dx}$$

Example 1:

$y = (x^2+1)(x+5)$

$\frac{dy}{dx} = (x^2+1)(1) + (x+5)(2x)$

$= x^2+1+2x^2+10x$

$= 3x^2+10x+1$

$y = (x^2+1)(x+5)$

$y = x^3+5x^2+x+5$

$\frac{dy}{dx} = 3x^2+10x+1$

Example 2: Find $\frac{dy}{dx}$ at $x=1$

$y = 6x^{-3}(x^2+2)$ [First $= 6x^{-3}$, Second $= (x^2+2)$

$$\begin{aligned}\frac{dy}{dx} &= (6x^{-3})(2x) + (x^2+2)(-18x^{-4}) \\ &= 12x^{-2} - 18x^{-2} - 36x^{-4} \\ &= -6x^{-2} - 36x^{-4} \\ &= -\frac{6}{x^2} - \frac{36}{x^4}\end{aligned}$$

at $x=1$

$$\begin{aligned}\frac{dy}{dx} &= \frac{-6}{1} - \frac{36}{1} \\ &= -42\end{aligned}$$

Note: You can extend this formula.

$$\frac{d(u_1 u_2 u_3)}{dx} = \frac{du}{dx} \cdot u_2 \cdot u_3 + u_1 \cdot \frac{du_2}{dx} \cdot u_3 + u_1 \cdot u_2 \cdot \frac{du}{dx}$$

3.) Quotient Rule: (den X der of num − num X der of den)
all divided by den. squared

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example 1: $y = \frac{(3x^2+4x-1)}{(2x+3)}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(2x+3)(6x+4) - (3x^2+4x-1)(2)}{(2x+3)^2} \\ &= \frac{12x^2+8x+18x+12-6x^2-8x+2}{(2x+3)^2} \\ &= \frac{6x^2+18x+14}{(2x+3)^2} \\ &= \frac{2(3x^2+9x+7)}{(2x+3)^2}\end{aligned}$$

Example 2: $y = \frac{x^3+3x+2}{3x^2-2x}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(3x^2-2)(3x^2+3) - (x^3+3x+2)(6x-2)}{(3x^2-2x)^2} \\ &= \frac{9x^4+9x^2-6x^3-6x - [6x^4-2x^3+18x^2-6x+12x-4]}{(3x^2-2x)^2}\end{aligned}$$

$$\frac{dy}{dx} = \frac{9x^4-6x^3+9x^2-6x-6x^4+2x^3-18x^2+6x-12x+4}{(3x^2-2x)^2}$$

$$\frac{dy}{dx} = \frac{3x^4-4x^3-9x^2-12x+4}{(3x^2-2x)^2}$$

4.) Application

Example: Suppose that the temperature T of food placed in a refrigerator drops according to the equation

$$T = 10\left(\frac{4t^2+16t+75}{t^2+4t+10}\right)$$

where t is the time in hours. What is the initial temperature of the food? Find the rate of change of T with respect to t when:

a.) $t=1$ b.) $t=3$

c.) $t=5$ d.) $t=10$

Initial Temperature is when $t=0$

$$\therefore\ T_{init} = 10\left(\frac{75}{10}\right) = 75^\circ$$

$$\frac{dT}{dt} = 10\left[\frac{(t^2+4t+10)(8t+16)-(4t^2+16t+75)(2t+4)}{(t^2+4t+10)^2}\right]$$

$$= 10\left[\frac{8t^3+16t^2+32t^2+64t+80t+160-8t^3-16t^2-32t^2-64t-150t-300}{(t^2+4t+10)^2}\right.$$

$$= 10\left[\frac{-70t-140}{(t^2+4t+10)^2}\right]$$

a.) at $t=1$

$$\frac{dT}{dt} = 10\left[\frac{-70(1)-140}{(1^2+4(1)+10)^2}\right]$$
$$= -9.33$$

b.) at $t=3$

$$\frac{dT}{dt} = 10\left[\frac{-70(3)-140}{(3^2+4(3)+10)^2}\right]$$
$$= -3.64$$

c.) at $t=5$

$$\frac{dT}{dt} = 10\left[\frac{-70(5)-140}{(5^2+4(5)+10)^2}\right]$$
$$= -1.61$$

d.) at $t = 10$ $\frac{dT}{dt} = \left[\frac{-70(10)-140}{(10^2+4(10)+10)^2}\right]$

$= -.37$

What is the limit of T as $t \to \infty$

$$\lim_{t\to\infty}\left[10\left[\frac{4t^2+16t+75}{t^2+4t+10}\right]\right]$$

$$= \lim_{t\to\infty} \frac{10\left(4+\frac{16}{t}+\frac{75}{t^2}\right)}{1+\frac{4}{t}+\frac{10}{t^2}}$$

$= 10 \cdot 4$

$= 40°$

Assignment: Calculus p109 (4, 5, 8, 10, 22, 25, 33, 36, 42)
APCalculus p109 (13, 15, 19, 22, 28, 29, 32, 33, 36, 42)

C. Implicit Differentiation and Fractional Powers - Section 2.3

Implicit Differentiation - This is differentiation when one variable is not given explicitly as a function of the other variable.

Rules for Implicit Differentiation

1.) Differentiate both sides of the equation with respect to x
2.) Collect all terms involving $\frac{dy}{dx}$ on the left side of the equation, and move all the others to the right side of the equation.
3.) Factor $\frac{dy}{dx}$ out of the left side of the equation.
4.) Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by the left-hand factor that does not contain $\frac{dy}{dx}$

Other definitions:

1.) Normal - the line perpendicular to the tangent of a curve at a given point P.

2.) Power Rule for Fractional Exponents:

$$\frac{d\,u^{P/Q}}{dx} = \frac{P}{Q}\,u^{(P/Q)-1}\,\frac{du}{dx}$$

Examples: Find $\frac{dy}{dx}$

1.)
$$\begin{aligned} y &= \sqrt[5]{(3x^2+4x+3)} \\ &= (3x^2+4x+3)^{1/5} \\ \frac{dy}{dx} &= \frac{1}{5}(3x^2+4x+3)^{-4/5}(6x+4) \\ &= \frac{2}{5}(3x+2)(3x^2+4x+3)^{-4/5} \end{aligned}$$

2.)
$$\begin{aligned} y &= \sqrt[7]{(x^2+3x+1)^4} \\ &= (x^2+3x+1)^{4/7} \\ \frac{dy}{dx} &= \frac{4}{7}(x^2+3x+1)^{-3/7}(2x+3) \\ &= 4/7\,(2x+3)(x^2+3x+1)^{-3/7} \end{aligned}$$

3.)
$$\begin{aligned} y &= \frac{(x^2+1)^3}{\sqrt{5x+1}} = \frac{(x^2+1)^3}{(5x+1)^{1/2}} \\ \frac{dy}{dx} &= \frac{(5x+1)^{1/2}(3)(x^2+1)^2(2x) - (x^2+1)^3\left(\frac{1}{2}\right)(5x+1)^{-1/2}(5)}{(5x+1)} \\ &= \frac{(5x+1)^{-1/2}(x^2+1)^2\left[6x(5x+1) - \frac{5}{2}(x^2+1)\right]}{(5x+1)} \\ &= \frac{\frac{1}{2}(x^2+1)^2\left[12x(5x+1) - 5(x^2+1)\right]}{(5x+1)^{3/2}} \\ &= \frac{\frac{1}{2}(x^2+1)^2\left[60x^2+12x-5x^2-5\right]}{(5x+1)^{3/2}} \\ &= \frac{(x^2+1)^2(55x^2+12x-5)}{2(5x+1)^{3/2}} \end{aligned}$$

4.) Do #3 as a product

$$y = (x^2+1)^3(5x+1)^{-1/2}$$

5.) $xy = 1$ ⇒ $y = x^{-1}$

$$\frac{dy}{dx} = -x^{-2}$$

$$= \frac{-1}{x^2}$$

$$x\frac{dy}{dx} + y(1) = 0$$

$$x\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = \frac{-y}{x}$$

from: $xy = 1$, $y = \frac{1}{x}$

$$\therefore \frac{dy}{dx} = \frac{-\frac{1}{x}}{x} = -\frac{1}{x^2}.$$

6.) $y^3 = 2x^2y$

$$3y^2\frac{dy}{dx} = 2x^2\frac{dy}{dx} + y(4x)$$

$$3y^2\frac{dy}{dx} - 2x^2\frac{dy}{dx} = 4xy$$

$$\frac{dy}{dx}(3y^2 - 2x^2) = 4xy$$

$$\frac{dy}{dx} = \frac{4xy}{(3y^2 - 2x^2)}$$

7.) Find the slope of the tangent line to the curve

$$x^2 + y^2 - 2x - 4y + 1 = 0$$

at the points: $(1,0)$, $(3,2)$, $(2, 2+\sqrt{3})$

* Note: This is not necessary, but by completing the square, we find the equation for this circle is:

$$(x-1)^2 + (y-2)^2 = 4$$

This tells us that the center is $(1,2)$, and the radius is 2.

Slope is the first derivative, therefore:

$$x^2 + y^2 - 2x - 4y + 1 = 0$$

$$2x + 2y\frac{dy}{dx} - 2 - 4\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} - 4\frac{dy}{dx} = -2x + 2$$

$$\frac{dy}{dx}(2y - 4) = -2(x-1)$$

$$\frac{dy}{dx} = \frac{-(x-1)}{(y-2)}$$

a.) at (1,0)

$$\frac{dy}{dx} = \frac{-(1-1)}{(0-2)} = \frac{0}{-2} = 0 \Rightarrow \text{horizontal line}$$

Note: The normal at this point is a vertical line $(x=1)$

b.) at (3,2)

$$\frac{dy}{dx} = \frac{-(3-1)}{(2-2)} = \frac{-2}{0} = \text{undefined} \Rightarrow \text{vertical line}$$

Note: The normal is a horizontal line $(y=2)$

c.) at $(2, 2+\sqrt{3})$

$$\frac{dy}{dx} = \frac{-(2-1)}{(2+\sqrt{3}-2)} = \frac{-1}{\sqrt{3}}$$

Note: The normal must be calculated

$$m_{tan} = -\frac{1}{\sqrt{3}} \Rightarrow m_{normal} = \sqrt{3}$$

$$\therefore \quad y-(2+\sqrt{3}) = \sqrt{3}\,(x-2)$$

$$y-2-\sqrt{3} = \sqrt{3}\,x - 2\sqrt{3}$$

$$y = \sqrt{3}\,x - \sqrt{3} + 2$$

Assignment: Calculus p116-117 (3, 6, 11, 17, 23, 28, 38, 44, 47, 48)
AP Calculus p116-117 (4, 10, 16, 22, 23, 29, 34, 38, 42, 46, 49, 51)

D. Linear Approximations and Differentials - Section 2.4

Linearization is a way of approximating the value of a function. The idea behind linearization is: The value of a function is approximately that of the tangent line.

Using the point-slope equation $(y-y_1)=m(x-x_1)$, and knowing the first derivative is the slope, we have this equation

$$y-f(a) = f'(a)(x-a)$$

This leads to the linearization formula of:

$$L(x) = f(a) + f'(a)(x-a)$$

Examples: Find the linearization and then estimate the given function value.

1.) $f(x) = x^4$, $a = 1$, $f(1.01)$

$f'(x) = 4x^3 \Rightarrow f'(a) = f'(1) = 4$, and $f(a) = 1$

$L(x) = f(a) + f'(a)(x-a)$

$L(x) = 1 + 4(x-1)$

$L(x) = 4x - 3$

$L(1.01) = 4(1.01) - 3$

$L(1.01) = 1.04$

Note: $f(1.01) = 1.04060401$

2.) $f(x) = x^2 + 2x$, $a = 0$, $f(.1)$

$f'(x) = 2x + 2 \Rightarrow f'(a) = f'(0) = 2$, and $f(a) = 0$

$L(x) = f(a) + f'(a)(x-a)$

$L(x) = 0 + 2(x-0)$

$L(x) = 2x$

$L(.1) = 2(.1)$

$L(.1) = .2$

Note: $f(.1) = (.1)^2 + 2(.1)$

$f(.1) = .21$

3 ways of describing the change in the value of a function

	True	Estimate
1.) Absolute change	Δf	df
2.) Relative change	$\frac{\Delta f}{f(x_0)}$	$\frac{df}{f(x_0)}$
3.) Percent change	$\frac{\Delta f}{f(x_0)} \times 100$	$\frac{df}{f(x_0)} \times 100$

Example: The radius of a circle is increased from 2.00 to 2.02 m.

a.) Estimate the resulting change in area

b.) Express the estimate as a percentage of the circle's original area.

$A = \pi r^2$, $dr = .02$, $r = 2$

a.) $dA = 2\pi r\,dr$

$dA = 2\pi(2)(.02)$

$dA = .08\pi$

b.) $\% = \frac{dA}{A} \times 100$, $A = \pi r^2 = \pi(2)^2 = 4\pi$

$= \frac{.08\pi}{4\pi} \times 100$

$= 2\%$ increase

Example: Find a) the change $\Delta y = f(a+\Delta x) - f(a)$, b) the value of the estimate $f'(a)\Delta x$, and c.) the error $\Delta y - f'(a)\Delta x$

$f(x) = x^3 - x$, $a = 1$, $\Delta x = .1$

a.) $\Delta y = f(1.1) - f(1) = [(1.1)^3 - 1.1] - [1^3 - 1]$

$= .231 - 0$

$\Delta y = .231$

b.) $f'(x) = 3x^2 - 1$

$dy = f'(1)\Delta x = [3(1)^2 - 1](.1)$

$= .2$

c.) error $= \Delta y - f'(a)\Delta x$

$= .231 - .2$

$= .031$

Formula: $(1+x)^k \doteq 1 + kx$

This approximation is used by science - look at p124-125 (Conversion of Mass to energy)

Lori's Problem

$$L(x) = f(a) + f'(a)(x-a)$$

$\sqrt[3]{998}$

$f(x) = \sqrt[3]{x}$, $a = 1000$

$f(x) = x^{1/3}$

$f'(x) = \frac{1}{3} x^{-2/3}$

$f(a) = f(1000) = \sqrt[3]{1000} = 10$

$f'(a) = f'(1000) = \frac{1}{3\sqrt[3]{(1000)^2}} = \frac{1}{3 * 100} = \frac{1}{300}$

$$L(x) = 10 + \frac{1}{300}(x - 1000)$$

$$= 10 + \frac{1}{300}x - \frac{10}{3}$$

$$= \frac{x}{300} + \frac{20}{3\cancel{0}}$$

$$L(998) = \frac{998}{300} + \frac{20}{3}$$

$$= 9.99\overline{3}$$

Section 2.4 page 126 (5, 8, 11, 14, 22, 25, 30, 36, 37)

5.) $f(x) = x^3 - x$, $a = 1$, $f(1.1)$
$f'(x) = 3x^2 - 1$, $f(1) = 0$, $f'(1) = 2$
$L(x) = f(a) + f'(a)(x-a)$
$= 0 + 2(x-1) = 2x - 2$
$L(1.1) = 2.2 - 2 = .2$

8.) $f(x) = (1+x)^{1/2}$, $a = 8$, $f(9.1)$
$f'(x) = \frac{1}{2(1+x)^{1/2}}$, $f(8) = 3$, $f'(8) = \frac{1}{6}$
$L(x) = 3 + \frac{1}{6}(x - 8) = 3 + \frac{1}{6}x - \frac{4}{3} = \frac{1}{6}x + \frac{5}{3}$
$L(9.1) = 3.183$

11.) $f(x) = (x^2+9)^{1/2}$, $a = -4$, $f(-4.2)$
$f'(x) = \frac{x}{(x^2+9)^{1/2}}$, $f(-4) = 5$, $f'(-4) = \frac{-4}{5}$ or $-.8$
$L(x) = 5 - .8(x+4) = 5 - .8x - 3.2 = -.8x + 1.8$
$L(-4.2) = -.8(-4.2) + 1.8 = 5.16$

14.) $(1+x)^3 \doteq 1 + 3x$

22.) $(1+x)^{-1/2} \doteq 1 - 1/2\,x$

25.) $f(x) = x^2 + 2x$, $a = 0$, $\Delta x = .1$
$f'(x) = 2x + 2$
a.) $\Delta y = f(.1) - f(0)$
$= .21 - 0 = .21$
b.) $f'(0)(.1) = (2)(.1) = .2$
c.) $.21 - .2 = .01$

a.) $\Delta y = f(a + \Delta x) - f(a)$
b.) $f'(a)\Delta x$
c.) $\Delta y - f'(a)\Delta x$ = a-b

30.) $f(x) = x^3 - 2x + 3$, $a = 2$, $\Delta x = .1$
$f'(x) = 3x^2 - 2$
a.) $\Delta y = f(2.1) - f(2) = 8.061 - 7 = 1.061$
b.) $f'(2)(.1) = 1.0$
c.) $1.061 - 1 = .061$

Example: Construct the linear approximation for:

$$3(1+x)^{1/3}$$
$$= 3\left(1+\tfrac{1}{3}x\right) = 3+x = x+3$$

Assignment: Calculus p126(7, 10, 12, 15, 19, 26, 28, 37)
AP Calculus p126(5, 8, 11, 14, 22, 25, 30, 36, 37)

E. The Chain Rule - Section 2.5

The chain Rule says: the derivative of a composite function is the product of their derivatives.
This formula will enable us to differentiate complicated functions using Known derivatives of simpler functions.

Chain Rule: h is the composite of $y=g(x)$ and $x=f(t)$
$(h = g\circ f)$

then

$$h'(t) = g'(f(t))\cdot f'(t)$$

or

$$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$$

There are two ways of doing this; 1.) use direct subs. (get the variables you want before you even start), and 2.) Find the der. of the first function, then get the der. of the second, and then mult. them together (this is using the chain rule).

Examples:

1.) Find $\frac{dy}{dt}$ if $y=x^3+3$ and $x=t^2+1$

Method # 1:

$$y = (t^2+1)^3+3$$
$$\frac{dy}{dt} = 3(t^2+1)^2(2t)$$

$$\frac{dy}{dt} = 6t(t^2+1)$$

Method #2:

$y = x^3+3$ $\qquad$ $x = t^2+1$

$\frac{dy}{dx} = 3x^2$ $\qquad$ $\frac{dx}{dt} = 2t$

$$\therefore \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$
$$= 3x^2 \cdot 2t$$
$$= 6t\,x^2$$

Since: $x = t^2+1$

$$\frac{dy}{dt} = 6t(t^2+1)^2$$

2.) Find $\frac{dy}{dt}$ if $y+4x^2=7$ and $x+\frac{5}{4}t=3$

Method #1:

$y = -4x^2+7$ and $x = -\frac{5}{4}t+3$

$$y = -4(-\tfrac{5}{4}t+3)^2+7$$
$$\frac{dy}{dt} = (-4)(2)(-\tfrac{5}{4}t+3)(-\tfrac{5}{4})$$
$$= 10(-\tfrac{5}{4}t+3)$$

Method #2:

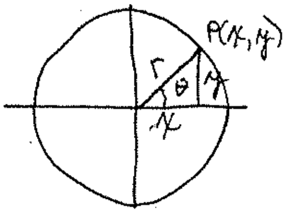
$y = -4x^2+7$ and $x = -\frac{5}{4}t+3$

$\frac{dy}{dx} = -8x$ $\qquad$ $\frac{dx}{dt} = -\frac{5}{4}$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dy}{dt}$$
$$= -8x \cdot (-5/4)$$
$$= 10x$$
$$= 10(-\tfrac{5}{4}t+3)$$

Assignment: Calculus p133-134 (3, 6, 7, 8, 12, 15, 18, 25)
AP Calculus p133-134 (2, 5, 7, 8, 11, 14, 19, 22, 26)

F. A Brief Review of Trigonometry - Section 2.6



Definitions of Trig Functions:

$\sin\theta = \frac{y}{r} = \frac{opp}{hyp}$ $\cos\theta = \frac{x}{r} = \frac{adj}{hyp}$

$\tan\theta = \frac{y}{x} = \frac{opp}{adj}$ $\cot\theta = \frac{x}{y} = \frac{adj}{opp}$

$\csc\theta = \frac{r}{y} = \frac{hyp}{opp}$ $\sec\theta = \frac{r}{x} = \frac{hyp}{adj}$

In terms of sin θ + cos θ

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\cot\theta = \frac{\cos\theta}{\sin\theta}$

$\sec\theta = \frac{1}{\cos\theta}$ $\csc\theta = \frac{1}{\sin\theta}$

2π radians = 360°
(Add the table on page 138 to students notebooks)
Know this table!

Graphing

$$f(x) = A\sin\left[\frac{2\pi}{B}(x-C)\right] + D$$

$|A|$ = amplitude
$|B|$ = period
C = horizontal shift
D = vertical shift

(Look at Example #2 on page 139 - interesting)

Definition: $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Note: Add the Definitions and Basic Identities on pages 141 and 142 to student's notebook. Know these!

Example: Use the formulas for $\sin(A+B)$ and $\cos(A+B)$ to derive the following:

1.) $\sin\left(x-\left(\frac{\pi}{2}\right)\right) = -\cos x$

$\sin\left(x+\left(-\frac{\pi}{2}\right)\right)$

$\sin x \cos\left(-\frac{\pi}{2}\right) + \cos x \sin\left(-\frac{\pi}{2}\right)$

$\sin x \cos\frac{\pi}{2} - \cos x \sin\frac{\pi}{2}$ $[\cos(-\theta) = \cos\theta]$

$\sin x(0) - \cos x(1)$ $[\sin(-\theta) = -\sin\theta]$

$-\cos x$

2.) $\sin\left(x+\frac{\pi}{2}\right) = \cos x$

$\sin x \cos\frac{\pi}{2} + \cos x \sin\frac{\pi}{2}$

$\sin x \; 0 + \cos x(1)$

$\cos x$

Assignment: Calculus p142-143 (2, 6, 10, 13, 14, 17, 23-25, 31-35)
AP Calculus p142-143 (5, 6, 11, 12, 15, 17, 23-25, 31-35)

G. Derivatives of Trigonometric Functions - Section 2.7

You must know these formulas:

1.) $\frac{d(\sin u)}{dx} = \cos u \frac{du}{dx}$

2.) $\frac{d(\cos u)}{dx} = -\sin u \frac{du}{dx}$

3.) $\frac{d(\tan u)}{dx} = \sec^2 u \frac{du}{dx}$

4.) $\frac{d(\sec u)}{dx} = \sec u \tan u \frac{du}{dx}$

5.) $\frac{d(\csc u)}{dx} = -\csc u \cot u \frac{du}{dx}$

6.) $\frac{d(\cot u)}{dx} = -\csc^2 u \frac{du}{dx}$

Examples: Find $\frac{dy}{dx}$

1.) $y = \sin x^2$

$\frac{dy}{dx} = \cos x^2 (2x) = 2x \cos x^2$

2.) $y = x^2 \tan x$

$\frac{dy}{dx} = x^2 \sec^2 x + \tan x \, (2x)$

$\frac{dy}{dx} = x^2 \sec^2 x + 2x \tan x$

Note: If you forget the rules (except sin + cos), put everything in terms of sin + cos.

3.) $y = \cot x$

$y = \frac{\cos x}{\sin x}$

$$\frac{dy}{dx} = \frac{\sin x(-\sin x) - \cos x \cos x}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x$$

Assignment: Calculus p149 (7, 12, 13, 16, 23, 24, 33, 38)
AP Calculus p149 (5, 10, 13, 16, 25, 26, 33, 36, 37)

H. Parametric Equations - Section 2.8

A parametric equation is an equation (a pair of equations) that define the motion of an object in a plane in terms of a third variable.

Equations: Given: $y = f(t)$ and $x = g(t)$

1.) $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

2.) $\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$

Examples: Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

1.) $x = 2t+3$, $y = t^2-1$

a.) This would be simple to get y in terms of x and solve. We will do that now.

$2t = x-3 \Rightarrow t = \frac{x}{2} - \frac{3}{2}$

$\therefore \quad y = \left(\frac{x}{2} - \frac{3}{2}\right)^2 - 1$

$\frac{dy}{dx} = (2)\left(\frac{x}{2} - \frac{3}{2}\right)\left(\frac{1}{2}\right)$

$= \frac{x-3}{2}$ (Note: this equals t)

$\frac{d^2y}{dx^2} = \frac{1}{2}$

b.) The second way is to use the formulas and method given for parametric equations.

Method (Finding --- $\frac{d^2y}{dx^2}$)

1.) Express $y' = \frac{dy}{dx}$ in terms of t.

2.) Differentiate y' with respect to t.

3.) Divide the result by $\frac{dx}{dt}$.

$x = 2t+3 \quad , \quad y = t^2 - 1$

$\frac{dx}{dt} = 2 \qquad \frac{dy}{dt} = 2t$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{2} = t$ (Note: $t = \frac{x-3}{2}$)

$y' = t$

$\frac{dy'}{dt} = 1$

$\therefore \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{1}{2}$

The answers are the same!

2.) $x = t^2 \quad , \quad y = 1 + t^3$

a.) $t = \sqrt{x} \Rightarrow y = 1 + x^{3/2}$

$\frac{dy}{dx} = \frac{3}{2} x^{1/2} = \frac{3}{2} t$

$\frac{d^2y}{dx^2} = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right) x^{-1/2}$
$= \frac{3}{4} x^{-\frac{1}{2}} = \frac{3}{4} t^{-1} = \frac{3}{4t}$

b.) $\frac{dx}{dt} = 2t \quad , \quad \frac{dy}{dt} = 3t^2$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{2t} = \frac{3}{2} t$

$\frac{dy'}{dt} = \frac{3}{2}$

$\therefore \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{3/2}{2t} = \frac{3}{4t}$

Note: You may be thinking - It is easier to find y in terms of x and then work with it. If you are thinking this, look at example #8 on page 153.

Example: Find $\frac{dy}{dx}$

$$x = 2\cos t \quad , \quad y = -2\sin t$$

$$\frac{dx}{dt} = -2\sin t \qquad \frac{dy}{dt} = -2\cos t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2\cos t}{-2\sin t} = -\frac{x}{y}$$

To check this, we can creat a Cartesian equation:

$$x^2+y^2 = (2\cos t)^2 + (-2\sin t)^2$$

$$x^2+y^2 = 4\cos^2 t + 4\sin^2 t$$

$$x^2+y^2 = 4$$

$$\therefore\ 2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Assignment: Calculus p154 (5, 13, 14, 26, 28, 32, 33, 36)
AP Calculus p154 (5, 13, 17, 24, 28, 32, 33, 36, 39)

I. Newton's Method – Section 2.9

Newton's Method is used for finding the root of an equation (approximate solution).
The formula is: Given $f(x) = 0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The steps are:
1.) Find $f'(x)$
2.) Pick an x_n (This is a guess of the solution)

3.) Subt. the x_n, value of $f(x_n)$ and the value of $f'(x_n)$ into the equation:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

4.) Continue this until you are satisfied with the answer. Note: use x_{n+1} for your new "guess" (x_n).

Example: I will state this problem 3 ways.

1.) Find the cube root of 2 using Newton's method.

2.) Find the root of the equation $f(x) = x^3 - 2 = 0$

Given: $f(x) = x^3 - 2$

$f'(x) = 3x^2$

(Guess an answer) $x_n = 1 \Rightarrow x_0 = 1$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = 1 - \frac{-1}{3} = 1.333\overline{3}$$

$$x_2 = 1.3\overline{3} - \frac{(1.3\overline{3})^3 - 2}{3(1.3\overline{3})^2} = 1.263888889$$

$$x_3 = 1.263888889 - \frac{(1.263888889)^3 - 2}{3(1.263888889)^2}$$

$$x_3 = 1.259933493$$

(Note: Calculator Answer = 1.25992105)

Most calculators use this method.

3.) You are at a programming contest, and the problem is to find the cube root of a number without the use of exponents. You have no idea how to do it, but then you remember Newton's method. (Note: To make the algebra easy to understand, I will do this for $\sqrt[3]{2}$ and then make it general.)

$$f(x) = x^3 - 2$$
$$f'(x) = 3x^2$$

$$\therefore x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^3 - 2}{3x_n^2}$$
$$= \frac{3x_n^3 - x_n^3 + 2}{3x_n^2}$$
$$= \frac{2x_n^3 + 2}{3x_n^2}$$
$$= \frac{2x_n}{3} + \frac{2}{3x_n^2}$$

The algebra is done. The only difference in finding $\sqrt[3]{2}$ and finding $\sqrt[3]{N}$ is the 2 in the term $2/3x_n^2$. We will make the term $N/3x_n^2$.

Now all we have to do is write the program, replacing $3x_n^2$ with $3*x_n*x_n$

The program (written for the Apple is:)

```
10 Home
20 Input "Enter # to find cube root of "; N
30 X = N/3 : Rem... This is my first guess at the answer
40 For I=1 to 50 : Rem... This is how many time
          the computer will go through the work
50 R=((2*X)/3)+(N/(3*X*X))
60 X=R
70 Next I
80 Print "The answer is"; R
```

Note: Look over the 5 remarks on pages 156 and 158.

Assignment: Calculus p158 (1,2,3,4,5) Look at #11
AP Calculus p158-159 (3,6,11)

J. Derivative Formulas in Differential Notation - Section 2.10

This section is very easy. All it is saying is that if you have a differential on one side of an equation, you must have a differential on the other side.
To do this, you simply find $\frac{dy}{dx}$ and then multiply both sides by dx.

Example: Find dy

$$y = x^2 + 2x + 1$$

$$\frac{dy}{dx} = 2x + 2$$

$$dy = (2x+2)\,dx$$

Assignment: Calculus p161 (1,2,3,6,7,14,15)
AP Calculus p161 (4,5,8,9,13,15)

K. Chapter 2 Test

Look at pages 161-165 Review Questions and Miscellaneous Problems.

III. Chapter 3 - Applications of Derivatives

A. Sketching Curves with the First Derivative - Section 3.1

Derivatives can be used to help us graph equations. The also can tell us where there are Local (or relative) Max and Min. points. They will also tell us where the curve is Increasing and Decreasing.

A function is increasing if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$. This says that a function is increasing when the slope of the tangent line is positive. (increasing if $f'(x) > 0$)

A function is decreasing if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$. This says that a function is decreasing when the slope of the tangent line is negative. (decreasing if $f'(x) < 0$).

If there is a value for which $f(x)$ or $f'(x)$ is undefined, then these values are critical values, and you must check both sides of these values. Also, $f'(x) = 0$ is a critical value, and you must check this point.

Examples: Find $\frac{dy}{dx}$ and the intervals of x-values on which $y = f(x)$ is increasing and decreasing. Sketch the curve, showing the points of transition between rising and falling portions. Find any local (relative) max and min values that the function has when $y' = 0$.

1.) $y = x^2$

$$\frac{dy}{dx} = 2x$$
$$0 = 2x$$
$$x = 0$$

Critical values

$x = 0$

$\frac{dy}{dx} = 2x$ — — | + +

0

Note:

Local Min

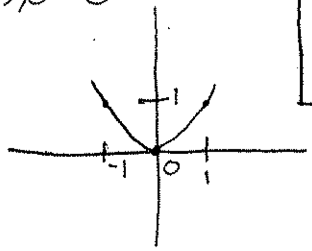
$\frac{dy}{dx}$ changes from − to + going from left to right.

Local Max

$\frac{dy}{dx}$ changes from + to − going from left to right.

a.) Critical values: $x=0$
b.) Increasing: $x>0$
c.) Decreasing: $x<0$
d.) Local Max: None
e.) Local Min: $x=0$

x	y
0	0
1	1
−1	1

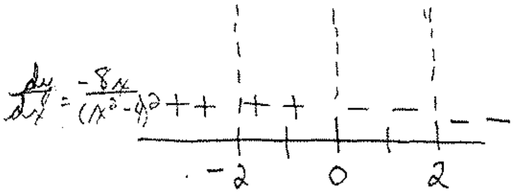


2.) $y=\frac{x^2}{x^2-4}$

$$\frac{dy}{dx}=\frac{(x^2-4)(2x)-x^2(2x)}{(x^2-4)^2}$$

$$=\frac{2x^3-8x-2x^3}{(x^2-4)^2}$$

$$\frac{dy}{dx}=\frac{-8x}{(x^2-4)^2}$$



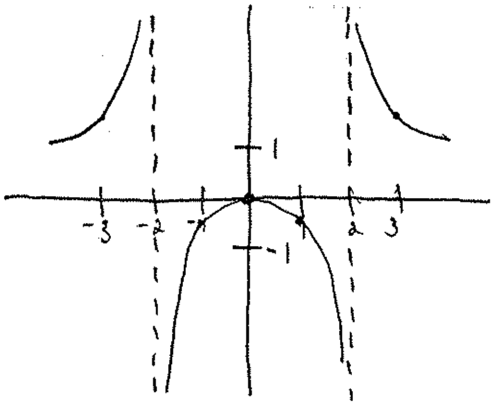
$$0=\frac{-8x}{(x^2-4)^2}$$

$-8x=0$

$x=0$, $x\neq\pm2$

a.) Critical values: $x=0, 2, -2$
b.) Increasing: $x<0$, $x\neq-2$
c.) Decreasing: $x>0$, $x\neq2$
d.) Local Max: $x=0$
e.) Local Min: None

x	y
0	0
1	-.33
-1	-.33
3	1.8
-3	1.8
1.999	-999.25
2.01	100.75

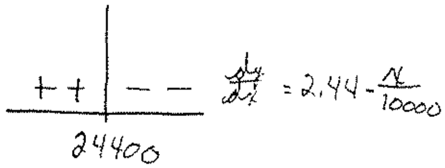


3.) A Fast-Food restaurant makes a profit P in selling x hamburgers, described by:

$$P = 2.44x - \frac{x^2}{20000} - 5000, \quad 0 \le x \le 35000$$

Find the intervals on which P is increasing or decreasing.

$$\frac{dP}{dx} = 2.44 - \frac{x}{10000}$$
$$0 = 2.44 - \frac{x}{10000}$$
$$\frac{x}{10000} = 2.44$$
$$x = 24400$$



Increasing: $0 \le x \le 24400$

Decreasing: $24400 < x \le 35000$

Assignment: Calculus p171 (1, 3, 8, 9, 15, 19)
AP Calculus p171 (4, 5, 8, 14, 15, 18, 19)

B. Concavity and Points of Inflection - Section 3.2

The second-derivative tells you 3 important items:

1.) The critical values to check for concave-up and concave-down. If the value of the second-derivative is positive, then the curve is concave-up. If it is negative, then the curve is concave-down.

2.) After setting the first-derivative to 0 and solving. Putting this value into the second-derivative will tell you if it is a local (relative) max. or min. point. If the second-derivative is positive, then that value is a local (relative) min. If it is negative, then the value is a local (relative) max.

3.) Points of Inflection - Setting the second-derivative to 0 and then solving, will tell you where the curve changes from concave-up to concave-down. You must also check any points where the second-derivative does not exist.

Examples: Find the following items:

a.) Where is the curve rising (increasing)
b.) Where is the curve falling (decreasing)
c.) Where is the curve concave-up
d.) Where is the curve concave-down
e.) The points of inflection
f.) Any local (relative) max. or min points
G.) Any asy.
H.) Sketch the curve

Examples

1.) $y = x - x^2$

$\frac{dy}{dx} = 1-2x$

$0 = 1-2x$

$2x = 1$

$x = \frac{1}{2}$

rise	fall
++	--
$\frac{1}{2}$	

$\frac{d^2y}{dx^2} = -2$

concave down
--

a.) $x < \frac{1}{2}$ rising

b.) $x > \frac{1}{2}$ falling

c.) Never concave up

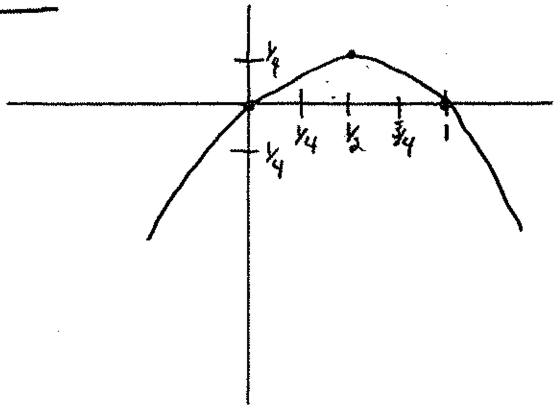
d.) always concave down

e.) NONE point of inflection

f.) $(\frac{1}{2}, \frac{1}{4})$ Max

g.) No asy. (None)

x	y
$\frac{1}{2}$	$\frac{1}{4}$
1	0
0	0



2.) $y = x - x^3$

$\frac{dy}{dx} = 1-3x^2$

$0 = 1-3x^2$

$3x^2 = 1$

$x^2 = \frac{1}{3}$

$x = \pm\sqrt{\frac{1}{3}} = \pm .577$

fall	rise	fall
--	++	--
	-.577	.577

a.) $-.577 < x < .577$

b.) $x < -.577$, $x > .577$

c.) $x < 0$

d.) $x > 0$

e.) $(0,0)$

f.) $m(-.577, -.385)$, $M(.577, .385)$

g.) No asy (None)

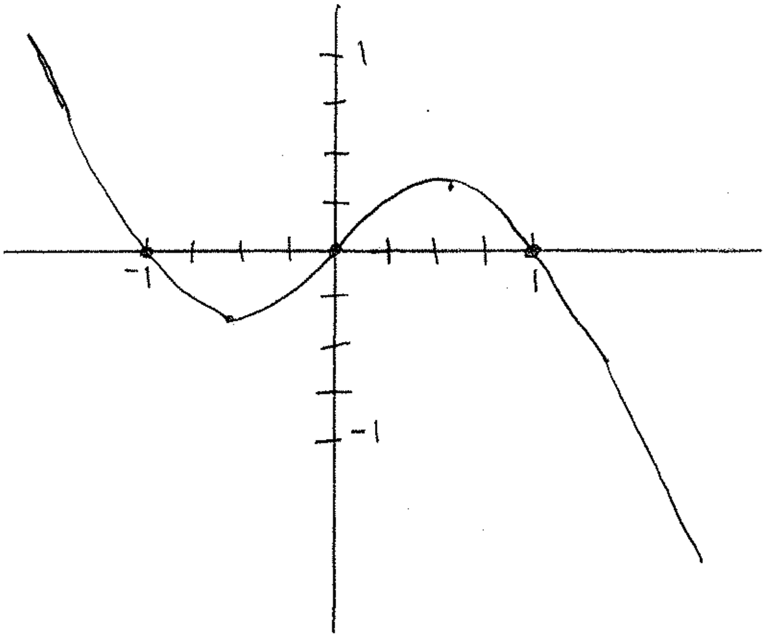
$\frac{d^2y}{dx^2} = -6x$

$0 = -6x$

$x = 0$

up	down
++	--
0	

x	y
0	0
1	0
-.577	-.385
.577	.385
-1	0



3.) $y = x(1+x^2)^{-1}$ $\quad y = \frac{x}{1+x^2}$

$$\frac{dy}{dx} = x(-1)(1+x^2)^{-2}(2x) + (1+x^2)^{-1}(1)$$
$$= (1+x^2)^{-2}[-2x^2+1+x^2]$$
$$= (1+x^2)^{-2}(1-x^2)$$

$$0 = \frac{1-x^2}{(1+x^2)^2}$$

$x^2 = 1$

$x = \pm 1$

fall	rise	fall
$--$	$++$	$--$
	-1	1

$$\frac{dy}{dx} = (1-x^2)(1+x^2)^{-2}$$

$$\frac{d^2y}{dx^2} = (1-x^2)(-2)(1+x^2)^{-3}(2x) + (1+x^2)^{-2}(-2x)$$
$$= (1+x^2)^{-3}(-2x)\,[(1-x^2)(2) + (1+x^2)]$$
$$= -2x(1+x^2)^{-3}(2-2x^2+1+x^2)$$
$$= -2x(1+x^2)^{-3}(3-x^2)$$

$$0 = \frac{-2x(3-x^2)}{(1+x^2)^3}$$

$-2x = 0 \qquad 0 = 3 - x^2$

$x = 0 \qquad x = \pm\sqrt{3}$

	down	up	down	up
	$--$	$++$	$--$	$++$
$(3-x^2)$	$--$	$++$	$++$	$--$
$-2x$	$++$	$++$	$--$	$--$
		$-\sqrt{3}$	0	$\sqrt{3}$

a) $-1 < x < 1$

b) $x < -1$, $x > 1$

c) $x > \sqrt{3}$, $-\sqrt{3} < x < 0$

d) $x < -\sqrt{3}$, $0 < x < \sqrt{3}$

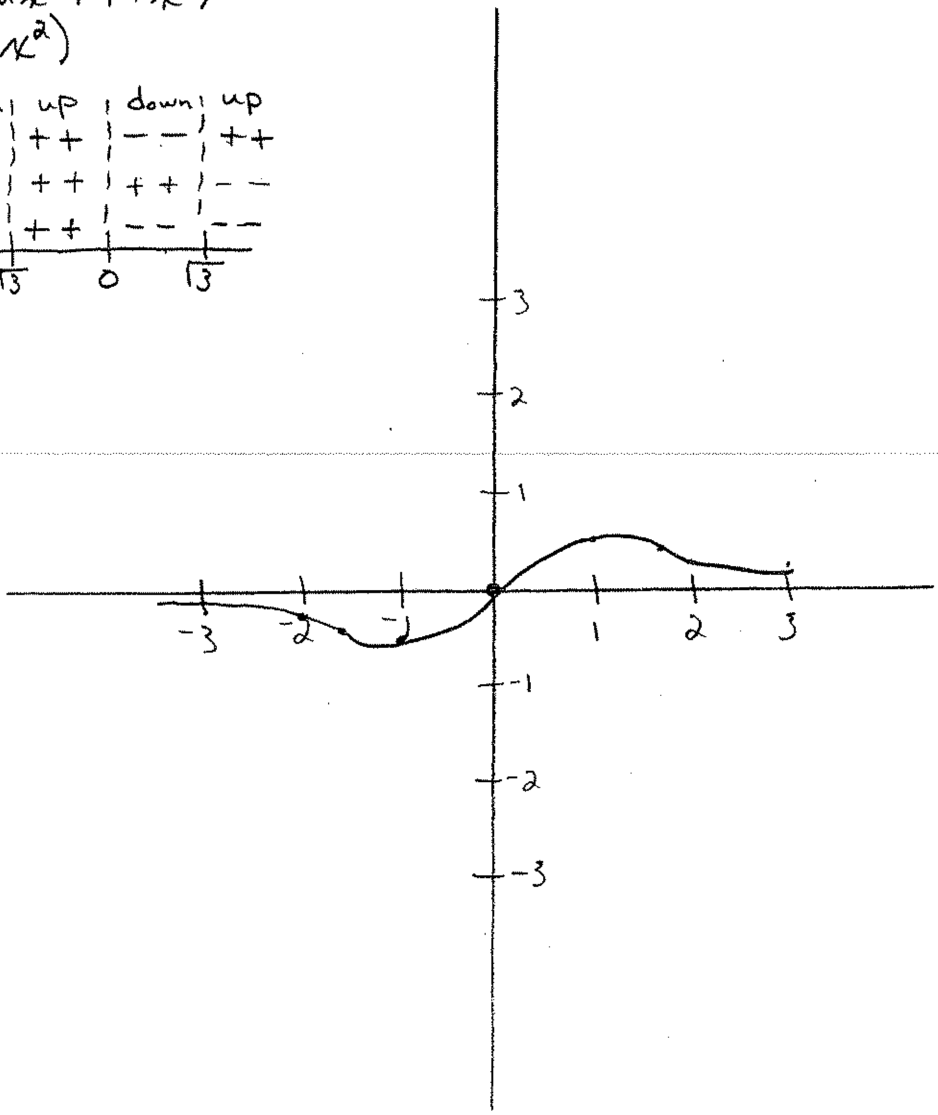
e) $(0,0)$, $(\sqrt{3}, \frac{\sqrt{3}}{4})$, $(-\sqrt{3}, -\frac{\sqrt{3}}{4})$

f.) $m(-1, -\frac{1}{2})$, $M(1, \frac{1}{2})$

g.) No asy (None)

$y = \frac{\sqrt{3}}{4}$

	x	y	
	0	0	
$\sqrt{3}$	1.73	.433	$\frac{\sqrt{3}}{4}$
$-\sqrt{3}$	-1.73	-.433	$-\frac{\sqrt{3}}{4}$
	-1	$-\frac{1}{2}$	
	1	$\frac{1}{2}$	
$\frac{-2}{5}$	-2	-.4	
	2	.4	
$\frac{-3}{10}$	-3	-.3	
	3	.3	



Examples

Calculus
Review for Quiz 5.4

44.

4.) $y = \frac{1}{4}x^4 - \frac{7}{2}x^2 - 6x + 1$

$\frac{dy}{dx} = x^3 - 7x - 6$

$= (x+1)(x^2 - x - 6)$

$= (x+1)(x+2)(x-3)$

$0 = (x+1)(x+2)(x-3)$

$x = -1, -2, 3$

	fall	rise	fall	rise
$(x-3)$	$- \ -$	$- \ -$	$- \ -$	$+ \ +$
$(x+2)$	$- \ -$	$+ \ +$	$+ \ +$	$+ \ +$
$(x+1)$	$- \ -$	$- \ -$	$+ \ +$	$+ \ +$

-2 -1 3

$\frac{dy}{dx} = x^3 - 7x - 6$

$\frac{d^2y}{dx^2} = 3x^2 - 7$

$0 = 3x^2 - 7$ $\left(x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$

$3x^2 = 7$

$x = \pm\sqrt{\frac{7}{3}}$

$x = \pm 1.53$

up	down	up
$+ \ +$	$- \ -$	$+ \ +$

-1.53 1.53

rise a.) $x > 3$, $-2 < x < -1$

fall b.) $x < -2$, $-1 < x < 3$

con up c.) $x > 1.53$, $x < -1.53$

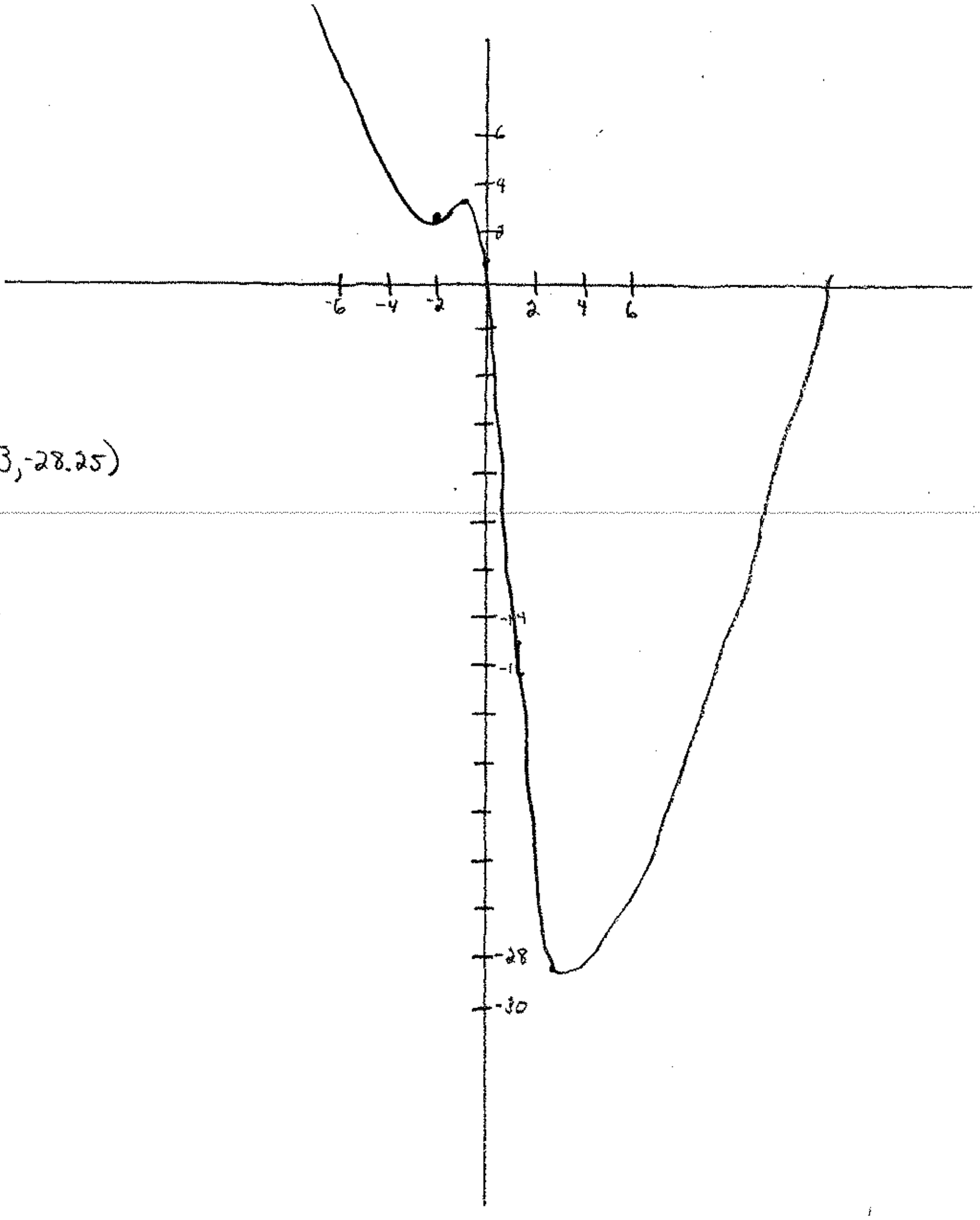
con down d.) $-1.53 < x < 1.53$

Infl. e) $(-1.53, 3.36)$, $(1.53, -15)$

max min f.) $M(-1, 3.75)$, $m(-2, 3)$, $m(3, -28.25)$

Asy. G.) None

x	y
-1	3.75
-2	3
3	-28.25
-1.53	3.36
1.53	-15.00
0	1



5.) $y = (x-3)^3 (x-2)^{-2}$

$$\frac{dy}{dx} = (x-3)^3(-2)(x-2)^{-3} + (x-2)^{-2}(3)(x-3)^2$$
$$= (x-3)^2(x-2)^{-3}\left[-2(x-3)+3(x-2)\right]$$
$$= (x-3)^2(x-2)^{-3}(-2x+6+3x-6)$$
$$= (x-3)^2(x-2)^{-3}(x)$$

$$0 = \frac{(x-3)^2(x)}{(x-2)^3}$$

	rise	fall	rise	rise
x	— — —	+ +	+ +	+ +
$x-3$	— —	— —	— —	+ +
$x-3$	— —	— —	— —	+ +
$(x-2)^3$	— —	— —	+ +	+ +

0 2 3

$x = 3, 0 \quad , x \neq 2$

$$\frac{dy}{dx} = x(x-3)^2(x-2)^{-3}$$

$$\frac{d^2y}{dx^2} = x(x-3)^2(-3)(x-2)^{-4} + (x-2)^{-3}\left[x(2)(x-3) + (x-3)^2(1)\right]$$
$$= -3x(x-3)^2(x-2)^{-4} + (x-2)^{-3}(2x^2-6x+x^2-6x+9)$$
$$= -3x(x-3)^2(x-2)^{-4} + (x-2)^{-3}(3x^2-12x+9)$$
$$= 3(x-2)^{-4}\left[-x(x-3)^2 + (x-2)(x^2-4x+3)\right]$$
$$= 3(x-2)^{-4}\left[-x(x^2-6x+9) + x^3-4x^2+3x-2x^2+8x-6\right]$$
$$= 3(x-2)^{-4}(-x^3+6x^2-9x+x^3-6x^2+11x-6)$$
$$= 3(x-2)^{-4}(2x-6)$$

	down	down	up
$6(x-3)$	— —	— —	+ +
$(x-2)^4$	+ +	+ +	+ +

2 3

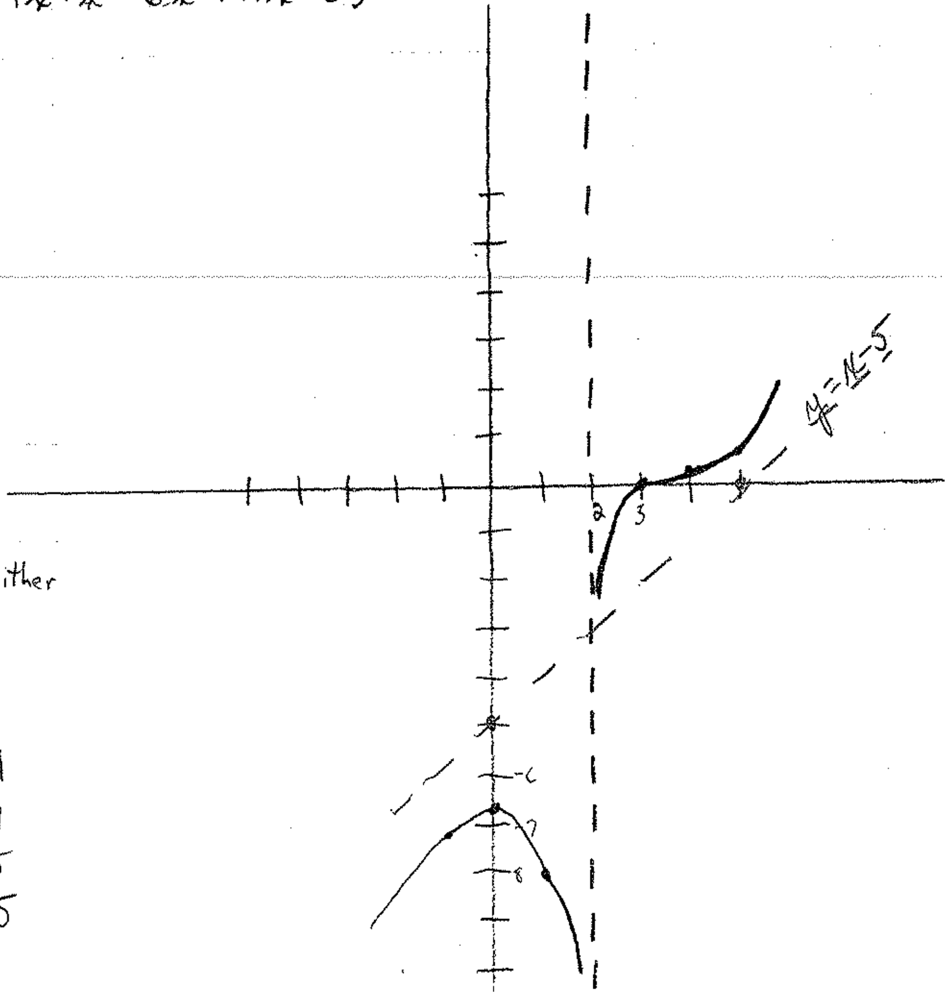
$$0 = \frac{6(x-3)}{(x-2)^4}$$

$x = 3 \quad , x \neq 2$

a.) $x<0$, $x>2$
b.) $0<x<2$
c.) $x>3$
d.) $x<3$
e.) (3,0)
f.) m(0,-6.75), (3,0) Neither
g.) $x = 2$

x	y
0	−6.75
3	0
2	—
1	−8

x	y
−1	−7.11
5	.89
4	.25
2.5	−.5



Assignment: Calculus p176(1,4,8,10,18,24,25,28,31)
AP Calculus p176(2,5,7,10,18,23,26,27,31)

C. Asymptotes and Symmetry - Section 3.3

In this section we will again be interested in graphing functions. We will cover asymptotes and symmetry to make our graphs more accurate.

Symmetry - we will only be interested in symmetry about the y-axis, and symmetry about the origin.

1.) Y-axis symmetry: this happens only when the function is even. This means $f(-x) = f(x)$. (Note: any function that has only even powers of x in it will be even. You still need to check).

2.) Origin symmetry: this happens only when a function is odd. This means $f(-x) = -f(x)$. You must always check this.

Asymptotes - We will be interested in horizontal, vertical, and oblique asymptotes.

1.) Horizontal asymptote: $y = b$ is a horizontal asymptote for $y = f(x)$ if either $\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$

2.) Vertical asymptote: $x=a$ is a vertical asymptote if either $\lim_{x\to a^+} f(x) = \pm\infty$ or

$$\lim_{x\to a^-} f(x) = \pm\infty$$

3.) Oblique asymptote: This occurs when you have a quotient of 2 polynomials that have no common factors, and the degree of the numerator is one greater than the denominator. To find this asymptote, divide the num. by the denominator, and the linear equation (without the remainder) is the oblique asymptote.

Example: Find the following:
- a.) symmetry
- b.) intercepts
- c.) asymptotes
- d.) slope at the intercepts
- e.) rise and fall
- f.) concave-up and concave-down
- g.) graph the function

$$y = \frac{x^2}{x-1}$$

a.) $f(-2) = \frac{(-2)^2}{-2-1} = \frac{4}{-3} = -\frac{4}{3}$

$f(2) = \frac{2^2}{2-1} = \frac{4}{1} = 4$

No symmetry to the origin or y-axis.

㊼

b.) $x=0 \Rightarrow y=\frac{0^2}{0-1}=0$

Note: y only $=0$ when $x=0$

$(0,0)$

c.) Horizontal: None

Vertical: $x=1$ $\qquad \lim_{x\to 1} f(x)=\infty$

Oblique:

$$\begin{array}{r} x+1+\frac{1}{x-1} \\ x-1\,\overline{)\,x^2\qquad} \\ \underline{x^2-x\quad} \\ x \\ \underline{x-1} \\ 1 \end{array}$$

$y=x+1$

d.) $y=x^2(x-1)^{-1}$

$y'=x^2(-1)(x-1)^{-2}+(x-1)^{-1}(2x)$

$=x(x-1)^{-2}\left[-x+2(x-1)\right]$

$=x(x-1)^{-2}(x-2)$

$\therefore\ y'(0)=\frac{0(0-2)}{(0-1)^2}=\frac{0}{1}=0$

e.) must check at $x=0,1,2$

x	$--$	$++$	$++$	$++$
$x-2$	$--$	$--$	$--$	$++$
$(x-1)^2$	$++$	$++$	$++$	$++$
y'	$++$	$--$	$--$	$++$

0 1 2

falling: $0\le x\le 2$

rising: $x\le 0,\ x\ge 2$

f.) $y' = (x^2-2x)(x-1)^{-2}$

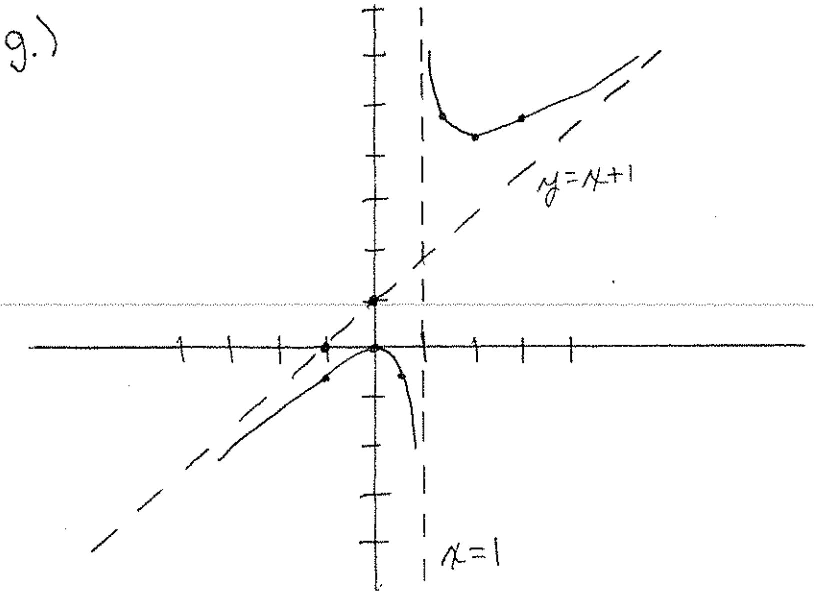
$$y'' = (x^2-2x)(-2)(x-1)^{-3} + (x-1)^{-2}(2x-2)$$
$$= (x-1)^{-3}\left[-2x^2+4x+(x-1)(2x-2)\right]$$
$$= (x-1)^{-3}\left[-2x^2+4x+2x^2-2x-2x+2\right]$$
$$= (x-1)^{-3}(2)$$
$$= \frac{2}{(x-1)^3}$$

y'' never $=0$, but it is undefined at $x=1$

2	$++$	$++$
$(x-1)^3$	$--$	$++$
y''	$--$	$++$

Concave-up: $x>1$
Concave-down: $x<1$

g.)



$y = \frac{x^2}{x-1}$

x	y
0	0
-1	$-\frac{1}{2}$
$\frac{1}{2}$	$-\frac{1}{2}$
2	4
3	$\frac{9}{2}$
$\frac{3}{2}$	$\frac{9}{2}$

Assignment: Calculus p184 (1-12, 17, 19, 28)
AP Calculus p184 (5-11, 15, 18, 23, 27)

D. Maxima and Minima: Theory - Section 3.4

There are many problems in math that deal with finding the "best" way to do something. These are called optimization problems. A large class of these problems can be reduced to finding the largest or smallest value of a function and determining where this value occurs. In this section, you will learn how to solve this type of problem.

Method - Given $y = f(x)$

1.) You must first find all the points (x-values) that may be either a local or absolute max. or min. value. This occurs when anyone of these are true:
    a.) at the endpoints of the domain. You must always check these values when you have a closed interval.
    b.) Find the first derivative and set it to zero. The x-value where the first derivative is zero could be the value you are looking for.
    c.) Any x-value where the first derivative is undefined.

2.) Test the points to find if they are max. or min. values. There are 2 tests for this.
    a.) Subs. the value that makes the first derivative zero into the second derivative.

If $f''(a) > 0$ then this is a local min. value.
If $f''(a) < 0$ then this is a local max. value.

b.) If the second derivative is too hard to find, or too hard to work with, you can check the sign change of the first derivative.

<u>Local Min.</u>: if $\frac{dy}{dx}$ changes from $-$ to $+$ going from left to right.

<u>Local Max.</u>: if $\frac{dy}{dx}$ changes from $+$ to $-$ going from left to right.

Examples: Find the critical points, determine whether these are local max. or min values, and if possible - find the absolute max. and min. values.

1.) $y = 4x^2 - 4x + 1$, $[0,1]$

$y' = 8x - 4$
$0 = 8x - 4$
$x = \frac{1}{2}$, other values to check are $x = 0, 1$

at $\frac{1}{2}$: $y = 4(\frac{1}{4}) - 4(\frac{1}{2}) + 1 = 0$
at 0: $y = 4(0) - 4(0) + 1 = 1$
at 1: $y = 4(1) - 4(1) + 1 = 1$

$y'' = 8$, always positive

Max value: $x = 0$, $x = 1$, $y = 1$ (absolute Max.)
Min value: $x = \frac{1}{2}$, $y = 0$ (absolute Min)

2.) $y = x^3 - 3x - 2 \quad , -\infty < x < \infty$

$y' = 3x^2 - 3$
$0 = 3x^2 - 3$
$0 = x^2 - 1$
$x = 1, -1$

at $x=1: y = 1 - 3(1) - 2 = -4$
at $x=-1: y = -1 + 3 - 2 = 0$

$y'' = 6x$
at $x=1$, y'' is $+$ $\Rightarrow$ min
at $x=-1$, y'' is $-$ $\Rightarrow$ max

Local Max at $x=-1$, $y=0$
Local Min at $x=1$, $y=-4$

Sign check (if y'' would be too hard)

3	$+\ +$	$+\ +\ +$	$+\ +$
$x+1$	$-\ -$	$+\ +\ +$	$+\ +$
$x-1$	$-\ -$	$-\ -\ -$	$+\ +$
$y' = 3x^2 - 3$	$+\ +$	$-\ -\ -$	$+\ +$

$-1 \qquad 1$

from left to right
at $x=-1$, changes from $+$ to $-$ $\Rightarrow$ Local Max.
at $x=1$, changes from $-$ to $+$ $\Rightarrow$ Local Min.

Assignment: Calculus p 191 (1, 4, 5, 12, 14, 18, 23)
AP Calculus p 191 (2, 3, 6, 10, 13, 17, 20, 23)

Procedure for solving Max/Min Problems

1. Read the problem.

2. Draw and label the necessary pictures.

3. Assign symbols and write down what is known.

4. Reduce the "Primary" equation to an equation having only one independent variable. Often other "Secondary" equations are useful.

5. Find the Max or Min by differentiating.

6. Check to determine if a max or min was found (use second derivative test or first derivative chart if second derivative test fails).

7. Check to make sure you answered the question.

E. Maxima and Minima: Problems - Section 3.5

In this section, you will use the methods developed to solve some applied optimization (Max.+Min.) problems.

Steps for solving Max-Min problems:

1.) Read the problem

2.) Draw and label the necessary pictures

3.) Name all the various quantities in the problem by letters, such as x, y, A, V, etc.

4.) Identify the variable to be maximized or minimized.

5.) Express the quantity to be max. or min. in terms of one or more quantities.

6.) By eliminating variables, express the quantity to be max. or min. as a function of one variable.

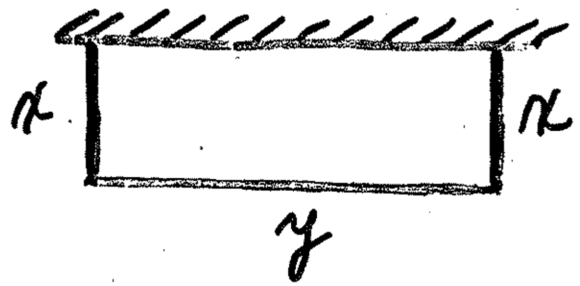
7.) Max. or min. the function - solve the problem and answer the questions.

Examples: Do the 6 examples in class, using the overhead projector.

Assignment: Calculus p199-201 (1, 2, 5, 8, 9, 11, 15, 18, 21, 24, 29, 30, 35, 38, 39)

AP Calculus p199-201 (1, 2, 5, 8, 9, 11, 15, 18, 21, 24, 29, 30, 35, 38, 39)

Example #1: A farmer has 80 running ft. of fencing available with which to construct a rectanglar enclosure along the side of his barn. What should the dimensions of the rectangle be if the area enclosed is to be a maximum?



$A = xy$

$80 = 2x + y$

$y = 80 - 2x$

$A = x(80-2x)$

$\frac{dA}{dx} = 80 - 4x$

$\frac{d^2A}{dx^2} = -4, < 0 \Rightarrow$ max.

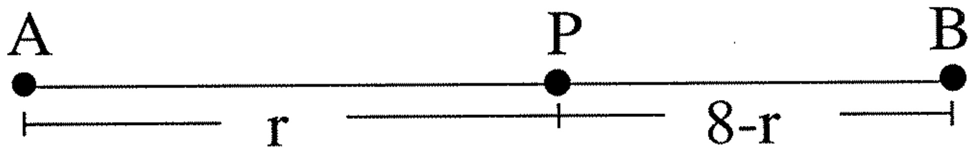
$0 = 80 - 4x$

$4x = 80$

$x = 20$ ft.

$y = 40$ ft.

Example #2: We are given heat sources at points A and B, 8 units apart, with the source at A twice as strong as that at B. If the heat received at a point is inversely proportional to the square of the distance from the heat source and directly proportional to the strength of that source, at what point on the line segment joining A to B will the heat received be a minimum?



$$H_A = \frac{2k}{r^2} \qquad H_B = \frac{k}{(8-r)^2}$$

$$H_T = \frac{k}{(8-r)^2} + \frac{2k}{r^2}$$

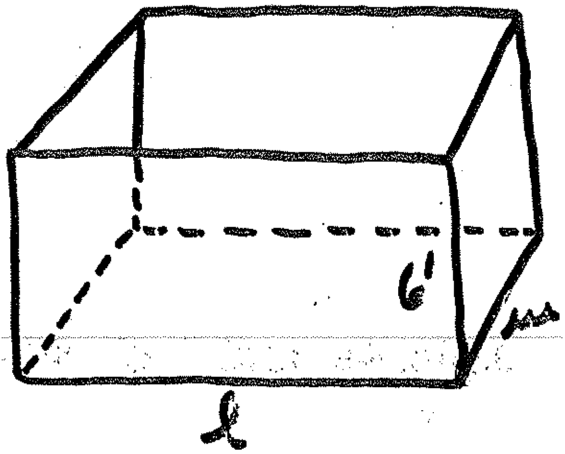
$$\frac{dH}{dr} = \frac{2k}{(8-r)^3} - \frac{4k}{r^3} \qquad \frac{d^2H}{dr^2} = \frac{6k}{(8-r)^4} + \frac{12k}{r^4} > 0 \Rightarrow \text{min}$$

$$0 = \frac{2k}{(8-r)^3} - \frac{4k}{r^3}$$

$$\frac{r^3}{(8-r)^3} = 2$$

$$r(1 + 2^{1/3}) = 8(2^{1/3}) \Rightarrow r = \frac{8(2^{1/3})}{1+(2)^{1/3}} = \frac{10.07}{2.26} = 4.46 \Rightarrow 8-r = 3.54$$

Example #3: An aquarium is to be 6 ft. high and is to have a volume of 750 cu. ft. The base, ends, and back are to be made of slate, but the front is to be made of a plate glass, which costs 1.5 times as much as the slate per sq. ft. What dimensions should be chosen to make the cost of raw materials a minimum?



$V = 750 \text{ ft}^3$

$V = 6 \times l \times w$

$750 = 6 \times l \times w$

$\Rightarrow w = \frac{750}{6l}$

$w = \frac{125}{l}$

$\text{Cost} = 6w + 6w + 6l + (1.5)(6)l + wl$

$= 12w + 15l + wl$

$\text{Cost} = 12\left(\frac{125}{l}\right) + 15l + \left(\frac{125}{l}\right)l$

$= \frac{1500}{l} + 15l + 125$

Example #3 (Page 2)

$$\frac{dc}{d\ell} = -\frac{1500}{\ell^2} + 15$$

$$0 = -\frac{1500}{\ell^2} + 15$$

$$\ell^2 = \frac{1500}{15}$$

$$\ell = 10 \text{ ft}$$

$$\frac{d^2c}{d\ell^2} = \frac{3000}{\ell^3} > 0 \Rightarrow \text{min.}$$

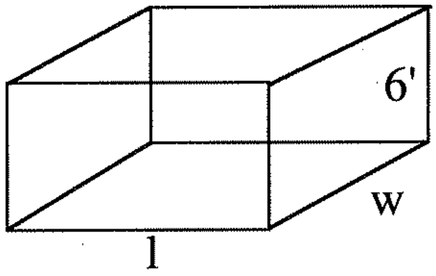
$$750 = 6 \times 10 \times w$$

$$w = \frac{750}{60}$$

$$w = 12.5 \text{ ft.}$$

$$\boxed{\begin{array}{c} h = 6' \\ \ell = 10' \\ w = 12.5' \end{array}}$$

Example #3: An aquarium is to be 6ft. high and is to have a volume of 750 cu. ft. The base, ends, and back are to be made of slate, but the front is to be made of a plate glass, which costs 1.5 times as much as the slate per sq. ft. What dimensions should be chosen to make the cost of raw materials a minimum?



$V=750\text{ft}^3$

$V=6lw$

$750=6lw$

$\Rightarrow w=\dfrac{750}{6l}$

$w=\dfrac{125}{l}$

$\text{Cost}=6w+6w+6l+(1.5)(6)l+wl$

$=12w+15l+wl$

$\text{Cost}=12(125l^{-1})+15l+(125l^{-1})l$

$=1500l^{-1}+15l+125$

$$\frac{dC}{dl}=\frac{-1500}{l^2}+15$$

$$0=\frac{-1500}{l^2}+15$$

$$l^2=\frac{1500}{15}$$

$l=10\text{ft}$

$$\frac{d^2C}{dl^2}=\frac{3000}{l^3}>0 \Rightarrow \text{min}$$

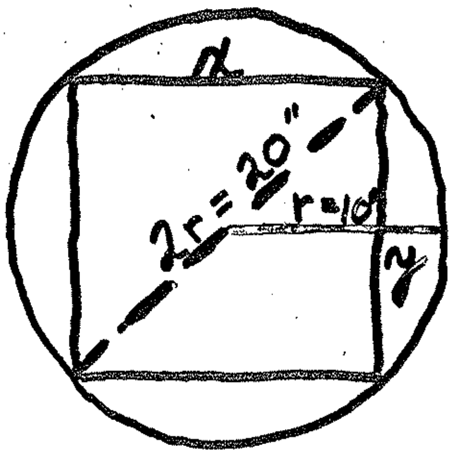
$750=6(10)(w)$

$$w=\frac{750}{60}$$

$w=12.5$ ft.

h= 6'
l= 10'
w= 12.5'

Example #4: Assume that the strength of a rectangular beam varies jointly as the width and the square of the depth. Which rectangular beam cut from a circular log of radius 10 in. will have the maximum strength?



Let x = width
y = depth

$S = kxy^2$
$S = kx(400 - x^2)$

$\frac{dS}{dx} = 400k - 3kx^2$
$0 = 400k - 3kx^2$
$3kx^2 = 400k$
$3x^2 = 400$
$x^2 = \frac{400}{3}$
$x = \sqrt{\frac{400}{3}}$
$x = \frac{20}{\sqrt{3}}$ or $\frac{20\sqrt{3}}{3} = 11.547$

$\frac{d^2S}{dx^2} = -6kx < 0 \Rightarrow$ max.

$20^2 = x^2 + y^2$
$y^2 = 20^2 - x^2$
$y^2 = 400 - x^2$

$400 = \frac{400}{3} + y^2$
$y^2 = 400 - \frac{400}{3}$

$y = \frac{20\sqrt{2}}{\sqrt{3}} = \frac{20\sqrt{6}}{3} = 16.329$

$$\boxed{\begin{array}{l} \text{depth} = \frac{20\sqrt{6}}{3} \\ \text{width} = \frac{20\sqrt{3}}{3} \end{array}}$$

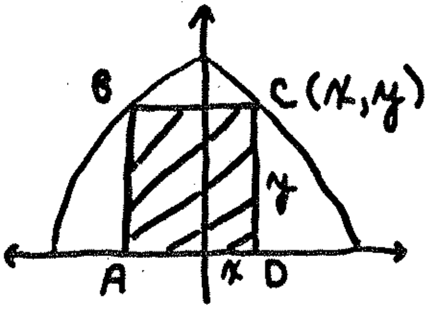
Example #5:

A rectangle ABCD with sides parallel to the coordinate axes is inscribed in the region enclosed by the graph of

$$y = -4x^2 + 4$$

and the x-axis. Find the x and y coordinates of C so that the area of rectangle ABCD is a maximum.

Optional Question: If point C moves along the curve with its x-coordinate increasing at the constant rate of 2 units per second. Find the rate of change of the area of rectangle ABCD when $x = 1/2$



$y = -4x^2 + 4$

$A = 2xy$

$A = 2xy$
$A = 2x(-4x^2+4)$
$A = -8x^3 + 8x$

$\frac{dA}{dx} = -24x^2 + 8$
$0 = -24x^2 + 8$
$24x^2 = 8$
$x^2 = 1/3$
$x = 1/\sqrt{3} = .577$

$y = -4x^2 + 4$
$= -4(.577)^2 + 4$
$y = 2.67$

$\boxed{(x, y) = (.577, 2.67)}$

$\frac{d^2A}{dx^2} = -48x, < 0 \Rightarrow$ max.

Optional Question:

$\frac{dx}{dt} = 2$, Find $\frac{dA}{dt}$ when $x = 1/2$

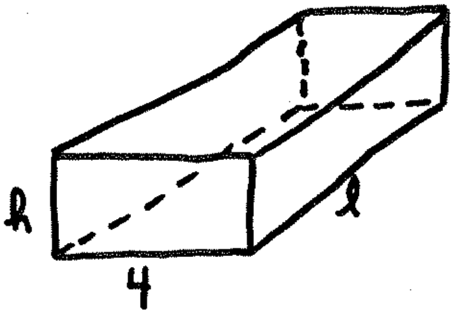
$A = -8x^3 + 8x$
$\frac{dA}{dt} = -24x^2 \frac{dx}{dt} + 8\frac{dx}{dt}$
$= -24(\frac{1}{2})^2(2) + 8(2)$

$= -12 + 16$

$\boxed{\frac{dA}{dt} = 4 \text{ units}^2/\text{sec}}$

Example #6:

A tank with a rectanglar base and rectanglar sides is to be open at the top. It is to be constructed so that its width is 4 meters and its volume is 36 cubic meters. If building the tank costs \$10.00 per square meter for the base and \$5.00 per square meter for the sides, what is the cost of the least expensive tank?



$V = wh\ell$
$V = 4h\ell$
$V = 36$
$36 = 4h\ell$
$9 = h\ell \Rightarrow \ell = \frac{9}{h}$

$C = 10(4\ell) + 5(h\ell + h\ell + 4h + 4h)$
$C = 40\ell + 10h\ell + 40h$

$C = 40(\frac{9}{h}) + 10h(\frac{9}{h}) + 40h$
$C = \frac{360}{h} + 90 + 40h$

$\frac{dc}{dh} = -\frac{360}{h^2} + 40$

$\frac{360}{h^2} = 40$

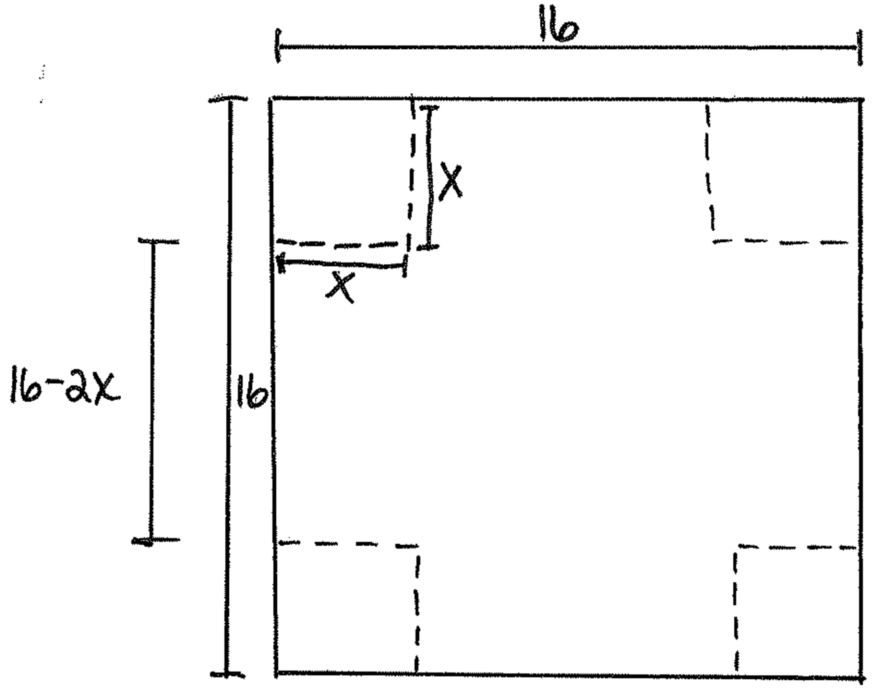
$40h^2 = 360$
$h^2 = 9$
$h = 3$

$\ell = \frac{9}{3} = 3$

$\frac{d^2c}{dh^2} = \frac{720}{h^3}, > 0 \Rightarrow \text{min.}$

$C = 40\ell + 10h\ell + 40h$
$C = 40(3) + 10(3)(3) + 40(3)$
$C = 120 + 90 + 120$
$C = 240 + 90$
$\boxed{C = \$330.00}$

Example 7: A square sheet of tin 16 inches on a side is to be used to make an open-top box by cutting a small square of tin from each corner and bending up the sides. How large a square should be cut from each corner to make the box have as large a volume as possible?



$V = l \cdot w \cdot h$

$V = (x)(16-2x)(16-2x)$

$V = x(256 - 64x + 4x^2)$

$V = 4x^3 - 64x^2 + 256x$

$dV/dx = 12x^2 - 128x + 256$

$0 = (6x-16)(2x-16)$

$6x - 16 = 0$

$6x = 16$

$x = 2\frac{2}{3} = \boxed{2.667}$

$2x - 16 = 0$

$2x = 16$

$\cancel{x = 8}$

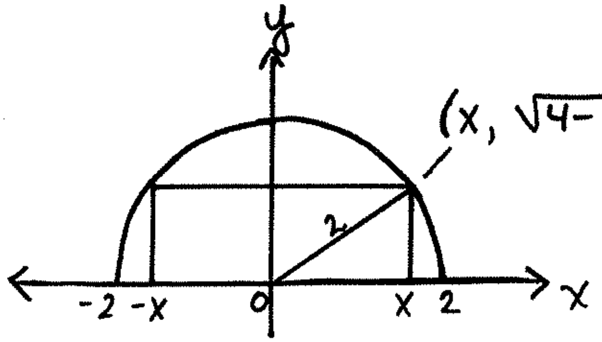
$\frac{d^2V}{dx^2} = 24x - 128$

at $x = 2.667$,

$< 0 \Rightarrow$ Max.

Example

A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have and what are its dimensions.



$x^2 + y^2 = 2^2$

$y^2 = 4 - x^2$

$y = \sqrt{4-x^2}$

Length : $2x$

Height : $\sqrt{4-x^2}$

Area: $2x \cdot \sqrt{4-x^2}$

$A = 2x(4-x^2)^{1/2}$

$$\frac{dA}{dx} = \frac{-2x^2}{\sqrt{4-x^2}} + 2\sqrt{4-x^2}$$

1^{st} derivative chart

$$\underset{\sqrt{2}}{+ + \mid - -}$$ MAXIMUM

$0 = -2x^2 + 2(4-x^2)$

$0 = -2x^2 + 8 - 2x^2$

$4x^2 = 8$

$x^2 = 2$

$x = \sqrt{2}$

$\text{Area} = 2x \cdot \sqrt{4-x^2}$

$= 2\sqrt{2} \cdot \sqrt{4-2}$

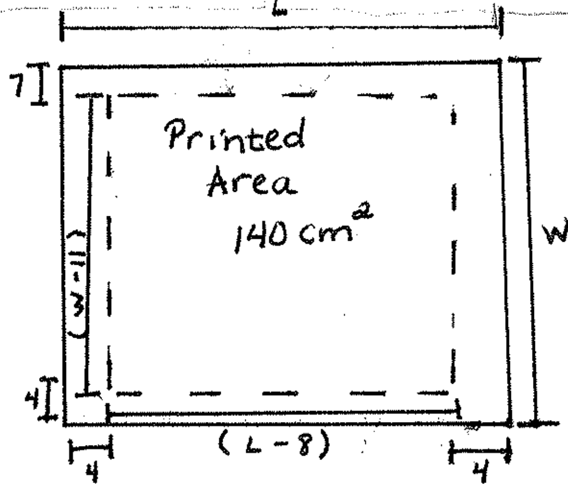
$= 2\sqrt{2} \cdot \sqrt{2}$

$= 2 \cdot 2$

$\text{Area} = 4$

Length : $\underline{2\sqrt{2}}$ Height : $\sqrt{4-2} = \underline{\sqrt{2}}$

Example: You are to design a poster to contain 140 square cm of ~~total~~ Printed area. It is to have margins of 7cm on the top and 4 cm on each of the other sides. What over all dimensions will ~~maximize the printed area.~~ min. the amount of paper used



$w = \frac{140}{l}$

Primary equation: P=(w-11)(L-8)
Secondary equation: A = Lw = 140
L= 140/ w
P=(w-11)(140/w - 8)
P=140 - 8w - 1540w^-1+ 88
P=228 - 8w - 1540w^-1
dp/ dw= -8 + 1540/ w^2
0 = -8w^2/ w^2 + 1540/ w^2
0 = -8w^2 + 1540
8w^2 = 1540
w^2 = 192.5
w = +13.9 , -13.9
d^2p/ dw^2 = -3080/ w^3
at 13.9 , < 0 = Max

L = 140/ 13.9
L = 10.1cm
w = 13.9cm

$A = (l+11)(w+8)$
$= (l+11)(\frac{140}{l}+8)$
$A = 140 + 8l + \frac{1540}{l} + 88$
$\frac{dA}{dl} = 8 - \frac{1540}{l^2}$
$0 = 8 - \frac{1540}{l^2}$
$8l^2 = 1540$
$l^2 = 192.5$
$l = 13.87$ cm
$w = 10.09$ cm

− − | + +
l 13.87

Answer: 18.09 cm x 24.87 cm

F. Related Rates of Change - Section 3.6

The rate at which something is changing is simply the derivative of the quantity with respect to time.

Method for solving Related Rate problems:

1.) Draw a picture and name the variables and constants. Use t for time and assume that all variables are differentiable functions of t.

2.) Write down any additional information. This should include numerical information and/or formulas.

3.) Write down what you are asked to find. Usually this is a rate, expressed as a derivative

4.) Write an equation the relates the variables. You may have to combine 2 or more equations to get a single equation that relates the variable whose rate you want to the variable whose rate you know.

5.) Differentiate your equation.

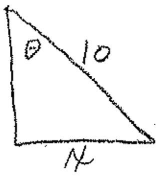
6.) Solve and answer the questions.

Examples:

There are 7 examples. Give as many as are needed to show the students the process for solving related rates. Note: The examples that are not used would make good Quiz questions.

related rate

A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 2ft/sec. How fast is the angle between the top of the ladder and the wall changing when the angle is $\frac{\pi}{4}$?



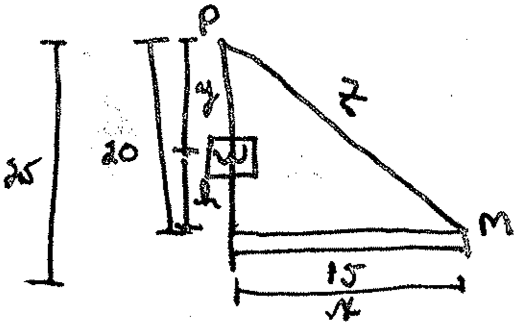
Find $\frac{d\theta}{dt} = ?$ $\frac{dx}{dt} = 2$

$\sin\theta = \frac{x}{10}$

Related Rates

Ex#1:

A rope running through a pulley at P, bearing a weight W at one end is being pulled at a rate of 6ft/sec by a man who is holding the other end of the rope in his hand 5 ft above the ground. The pulley is 25 ft above the ground, the rope is 45 ft long, and at a given instant the man is 15 ft away from the ⊥ of the weight. How fast is the weight being raised at this particular instant? (Find $\frac{dh}{dt}$)



Given: $y+z=45$

$h+y=20$

$20^2+x^2=z^2$

At the given instant: $x=15$ $\frac{dx}{dt}=6$
(at $t=0$)

Key idea: Get an equation that relates x to h

$z=45-y$

~~$h=20-y$~~ $y=20-h$

$\therefore z=45-(20-h)$

$\Rightarrow z=25+h$

This yields the equation

$$20^2+x^2=(25+h)^2$$

$$\frac{d(20^2+x^2)}{dt}=\frac{d(25+h)^2}{dt}$$

$$2x\frac{dx}{dt}=2(25+h)\frac{dh}{dt} \Rightarrow \frac{dh}{dt}=\frac{x}{(25+h)}\frac{dx}{dt}$$

subs. $2(15)(6)=2(25+h)\frac{dh}{dt}$

$(20^2+15^2)=(25+h)^2$

$625=(25+h)^2$

$25+h=25$

$$\frac{dh}{dt}=\frac{(15)(6)}{(25+h)}$$

$$\boxed{\frac{dh}{dt}=\frac{90}{25}=3.6\text{ ft/sec.}}$$

RELATED RATES

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Example #2: A conical icicle, whose height is always 12 times the radius of its base, is being formed by the dripping of water. If the volume is increasing at the rate of 1 cubic cm. per hour, at what rate is the height increasing when the height is 8 cm. ?

FIND: $\frac{dh}{dt}$

GIVEN: $r = \frac{1}{12} \cdot h$

$V = \frac{1}{3} \pi r^2 h$

at a given instant: $\frac{dv}{dt} = 1; \quad h = 8$

subs: $V = \frac{1}{3} \pi (\frac{1}{12} \cdot h)^2 h \implies V = \frac{1}{432} \pi h^3$

$$\frac{dv}{dt} = \frac{d(\frac{1}{432} \pi h^3)}{dt}$$

$$\frac{dv}{dt} = \frac{1}{144} \pi h^2 \frac{dh}{dt} \implies \frac{dh}{dt} = \frac{144}{\pi h^2} \frac{dv}{dt}$$

subs: $\frac{dh}{dt} = \frac{144}{(3.14)(8)^2}(1)$

$\frac{dh}{dt} = 0.72$ cm/hr.

RELATED RATES

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EXAMPLE #3: A dock stands 8 ft. above the deck of a boat. The boat is being pulled into the dock by means of a rope attached to the deck at the front of the boat. If 2 ft of rope is drawn in each minute, at what rate is the boat moving toward the dock when the boat is 15 ft. away?

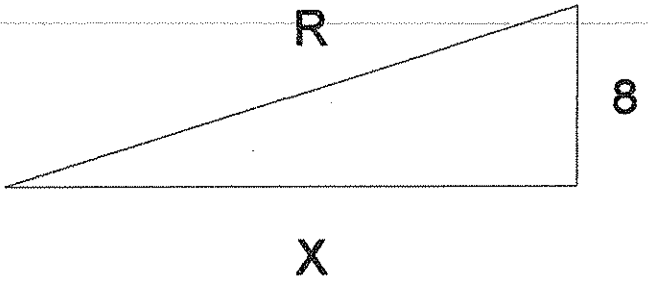
GIVEN: $x^2 = r^2 - 8^2$

At a given instant: $\frac{dr}{dt} = 2$, $x = 15$

FIND: $\frac{dx}{dt}$

$$\frac{d(x_2)}{dt} = \frac{d(r2-64)}{dt}$$

$$2x\frac{dr}{dt} = 2r\frac{dx}{dt} - 0$$



at x = 15: $15^2 = r^2 - 64$

$$r^2 = 225 + 64$$

$$r^2 = 289$$

$$r = 17$$

$$2(15)\frac{dx}{dt} = 2(17)(2)$$

SUBS: $\frac{dx}{dt} = \frac{34}{15} = 2.27$ ft./min.

RELATED RATES

Sec. 5.7

Example #4: A person 1.8m tall is walking away from a lamppost 5m high at a rate of 2 m/sec. At what rate is the end of the person's shadow moving away from the lamppost?

Given: $\frac{dx}{dt}$ = 2 m/sec.

Find: $\frac{dy}{dt}$

From geometry you know that △ABC and △DEC are similar, therefore $\frac{DC}{DE} = \frac{AC}{AB}$

This will give us an equation relating X and Y.

Therefore : $\frac{y-x}{1.8} = \frac{y}{5}$

Solution:

$\frac{y-x}{1.8} = \frac{y}{5}$

$5(y-x)=1.8y$

$5y-5x=1.8y$

$3.2y=5x$

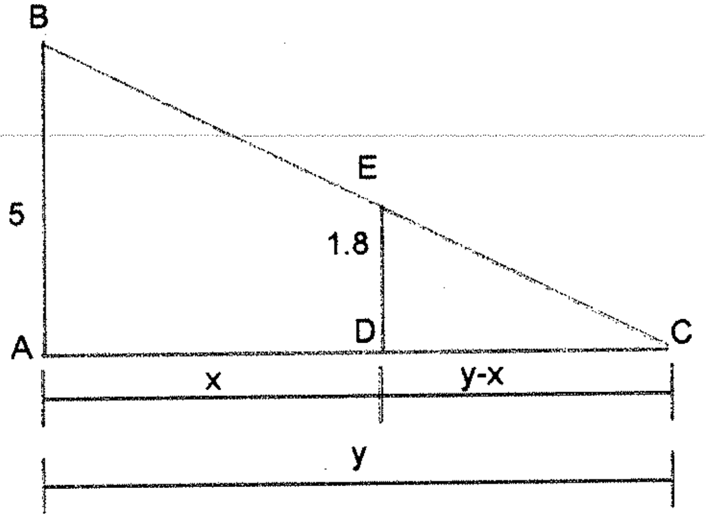
$3.2\frac{dx}{dt} = \frac{dy}{dt}$

$\frac{dy}{dt} = \frac{5}{3.2}\frac{dx}{dt}$

$\frac{dy}{dt} =(\frac{5}{3.2})(\frac{2}{1})$

$\frac{dy}{dt} = \frac{10}{3.2}$

$\frac{dy}{dt} =$ 3.125 m/sec.



RELATED RATES

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EXAMPLE #5: When a gas is compressed adiabatically (with no gain or loss of heat) it satisfies the equation $PV^{1.4}=k$, where k is a constant. At a given instant the pressure P is 40 atmospheres and the volume V is 28 liters and is decreasing at a rate of 2 liters/min. At what rate is the pressure changing?

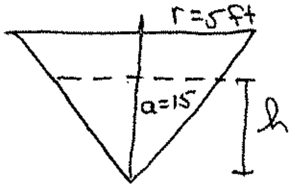
GIVEN: $PV^{1.4}=k$, P=40 atm, V=28l, $\frac{dv}{dt}=-2$l/min.
FIND: $\frac{dp}{dt}$

$$PV^{1.4} = k$$
$$P(1.4)V^{.4}\frac{dv}{dt} + V_{1.4}\frac{dp}{dt} = 0$$
$$V^{1.4}\frac{dp}{dt} = -1.4PV_{.4}\frac{dv}{dt}$$
$$(28)^{1.4}\frac{dp}{dt} =(-1.4)(40)(28)^{.4}(-2)$$

$$106.175\frac{dp}{dt} = 424.7$$
$$\frac{dp}{dt} =\frac{424.7}{106.175}$$
$$\frac{dp}{dt} =4 \text{ atm/min.}$$

Related Rates

Example #6: Consider a cistern in the shape of an inverted right circular cone where the altitude is 15 ft. and the radius is 5 ft. If water is being pumped in at a rate of 12 ft³/min, at what rate is the depth h of the water increasing when the depth is 4 ft.



$V=\frac{1}{3}\pi r^2 h$, $\frac{dV}{dt}=12\,ft^3/min$, $h=4\,ft$, $\frac{dh}{dt}=?$

using similar triangles :

$$\frac{\text{radius}}{\text{depth}}=\frac{5}{15}=\frac{1}{3} \Rightarrow r=\frac{1}{3}h$$

$$\therefore V=\frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h$$

$$V=\frac{1}{27}\pi h^3$$

$$\frac{dV}{dt}=\frac{1}{9}\pi h^2\frac{dh}{dt}$$

$$12=\frac{1}{9}\pi(16)\frac{dh}{dt}$$

$$\frac{dh}{dt}=\frac{(12)(9)}{(16)(\pi)}$$

$$\frac{dh}{dt}\doteq 2.15\ ft/min$$

Practice Problem: Same as example #6 except:
alt. = 10 ft, radius = 5 ft, rate = 2 ft/min (rising),
depth = 4 ft.

Question: At what rate is the water entering the cistern?

$$V=\frac{1}{3}\pi r^2 h$$

$$r=\frac{1}{2}h$$

$$\therefore V=\frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h$$

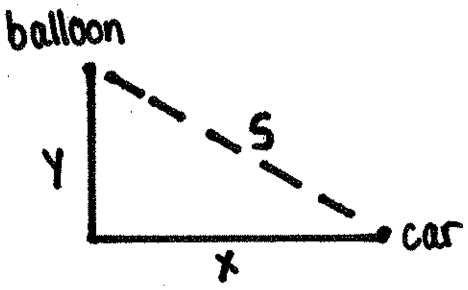
$$V=\frac{1}{12}\pi h^3$$

$$\frac{dV}{dt}=\frac{1}{4}\pi h^2\frac{dh}{dt}$$

$$\frac{dV}{dt}=\frac{1}{4}\pi(16)(2)$$

$$=8\pi\ ft^3/min$$

Example #7: A child riding in a car accidentally releases a helium balloon that rises vertically at 60ft./sec. while the car continues to travel at 80 ft./sec. on the straight road. How fast are the car and balloon separately after 2 sec.; after t seconds.



$s^2 = x^2 + y^2, \frac{dx}{dt} = 80, \frac{dy}{dt} = 60$, Find $\frac{ds}{dt}$ at t=2

at t=2, x=160,y=120 => $s^2 = 160^2 + 120^2$

$s = 200$

$$s^2 = x^2 + y^2$$
$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$
$$s\frac{ds}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

$$200\frac{ds}{dt} = 160(80) + 120(160)$$

$$\frac{ds}{dt} = 100 \text{ ft./sec.}$$

After t seconds, x= 80t, g= 60t => s= 100t

$$100t\frac{ds}{dt} = 80t\,(80) + 60t\,(60)$$
$$\frac{ds}{dt} = 100\text{ft./sec.}$$

We conclude that s is increasing at the constant rate of 100ft./sec.

Assignment: Calculus ~~pps~~ 207-209 (2, 3, 6, 8, 9, 10, 13, 15, 16, 19, 22, 28)
AP Calculus ~~pps~~ 207-209 (2, 3, 6, 8, 9, 10, 13, 15, 16, 19, 22, 28)

G. The Mean Value Theorem - Section 3.7

The Mean Value Theorem is sometimes confused with Rolle's Theorem.

Note: Michel Rolle (1652-1719) disbelieved in Calculus and tried to disprove it. It is funny that he is best remembered for his contribution to Calculus.

Rolle's Theorem: If $f(a) = f(b) = 0$ and $f(x)$ is continuous on $[a, b]$ and is differentiable on (a, b) then there is at least on number C between a+b where

$$f'(C) = 0$$

One interesting application of this is in locating solutions for equations.

1.) If $f(x)$ is continuous on $[a, b]$ and is differentiable on (a, b);
2.) $f(a)$ and $f(b)$ have opposite signs, and
3.) f' is never zero between a+b

then $y = f(x)$ has only one zero between a+b. This is ossured by the Intermediate Value Theorem of section 1.11.

Example: Show that the equation

$x^4+3x+1=0$, $[-2,-1]$

has exactly on solution onthe given interval.

$f(x)=x^4+3x+1$

$f(-2)=11$, $f(-1)=-1$

$f'(x)=4x^3+3$

$0=4x^3+3$

$4x^3=-3$

$x^3=-\frac{3}{4}$

$x=-\sqrt[3]{\frac{3}{4}}$, $-(\frac{3}{4})^{1/3}>-1$

∴ there is exact'ly one solution

Mean Value Theorem: If $y=f(x)$ is continuous on $[a,b]$ and differentiable on (a,b) then

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

Example: Using the Mean Value theorem, find c.

$f(x)=x^2+2x-1$; $a=0$, $b=1$

$f'(x)=2x+2 \Rightarrow f'(c)=2c+2$

$f(a)=f(0)=0^2+2(0)-1=-1$

$f(b)=f(1)=1^2+2(1)-1=2$

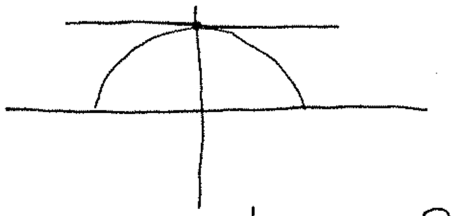
∴ $2c+2=\frac{2-(-1)}{1-0}$

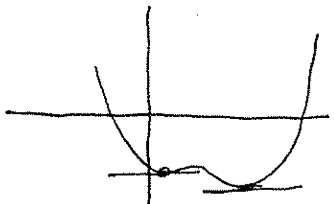
$2c+2=3$

$2c=1$

$c=1/2$

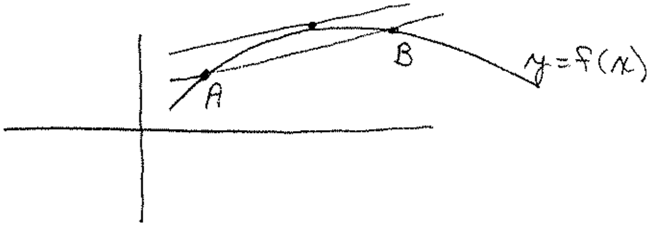
Rolle's Theorem says "graphicly" that there is at least one place where the tangent line to the curve is horizontal. ("well-behaved" curve)





This is a special case of the Mean Value Theorem.

The Mean Value theorem says that there must be at least on place where the tangent line to the curve is parallel to the secant line joining A and B.



Assignment: Calculus p215 (2,3,8,9)
AP Calculus p215 (2,3,8,9)

H. Indeterminate Forms and L'Hopital's Rule - Section 3.8

In this section we will learn how to find the limit if it is in the form $\frac{0}{0}$, or some other indeterminate form.

The rule is called L'Hopital's Rule. I was published by him in the first textbook ever published (1696) on differential calculus. It is now believed that it was given to him by John Bernoulli.

This rule is very easy to understand.

Given that x_0 could be any of the following:

$a, a^+, a^-, +\infty, -\infty$

then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}$$

How to use L'Hopital's Rule

1.) Chech that $\lim \frac{f(x)}{g(x)}$ is an indeterminate form. If it is not, you can not use the rule.

2.) Differentiate f and g separately

3.) Find the $\lim \frac{f'(x)}{g'(x)}$. If this limit is finite, $+\infty$, or $-\infty$, then it is equal to $\lim \frac{f(x)}{g(x)}$

4.) Stop differentiating as soon as you get something other than $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Examples: Find the limits

1.) $\lim\limits_{x \to 2} \frac{x^2-4}{x-2}$

Using the method we had before

$$\lim_{x \to 2} \frac{(x+2)(x-2)}{(x-2)} = \lim_{x \to 2} (x+2) = 4$$

Using L'Hopital's Rule

$f(x) = x^2 - 4 \Rightarrow f'(x) = 2x$

$g(x) = x - 2 \Rightarrow g'(x) = 1$

$\therefore \lim_{x \to 2} \frac{f'(x)}{g'(x)} = \lim_{x \to 2} \frac{2x}{1} = 4$

2.) $\lim_{x \to 0} \frac{\sin 2x}{x}$

$f(x) = \sin 2x \Rightarrow f'(x) = 2\cos 2x$

$g(x) = x \Rightarrow g'(x) = 1$

$\therefore \lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{2\cos 2x}{1} = \frac{2(1)}{1} = 2$

3.) $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$ This is in the form $\infty - \infty$, which we can not use. We need it in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. We will use a little algebra, and we get:

$\lim_{x \to 0^+} \frac{\sin x - x}{x \sin x}$ This is now $\frac{0}{0}$.

$f(x) = \sin x - x \Rightarrow f'(x) = \cos x - 1$

$g(x) = x \sin x \Rightarrow g'(x) = x\cos x + \sin x$

$\lim_{x \to 0^+} \frac{\cos x - 1}{x\cos x + \sin x}$ This is still $\frac{0}{0}$, so we do it again

$f(x) = \cos x - 1 \Rightarrow f'(x) = -\sin x$

$g(x) = x\cos x + \sin x \Rightarrow g'(x) = \cos x + \cos x - x\sin x$

$\lim_{x \to 0^+} \frac{-\sin x}{2\cos x - x\sin x} = \frac{-0}{2} = 0$

Assignment: Calculus p221 (1, 2, 4, 5, 8, 10, 14, 16, 24)
AP Calculus p221 (3, 6, 7, 10, 14, 15, 20, 22, 23, 24)

I. Quadratic Approximations and Approximation Errors: Extend the Mean Value Theorem - Section 3.9

In this section we will be mainly interested in the Quadratic Approximation of $f(x)$ near a given value $x=a$, and the approximation error.
Note: Look at table 3.2 on the top of page 229.

Quadratic Approximations are more precise than Linear Approximations.

Quadratic Approximation: (near) $x=a$ of $f(x)$

$$Q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

The max. error for this approximation is:

$$|e_2(x)| \le \frac{1}{6} M |x-a|^3$$

where M is the upper bound for the values of $|f'''(x)|$.

Example: Find the Quadratic Approximation and the Approximation error for:

$(1+x)^{1/2}$ near $x=0$, How accurate if $|x| \le .1$

$$f(x) = (1+x)^{1/2}$$
$$f'(x) = \frac{1}{2}(1+x)^{-1/2}$$
$$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)(1+x)^{-3/2} = -\frac{1}{4}(1+x)^{-3/2}$$
$$f'''(x) = -\frac{1}{4}\left(-\frac{3}{2}\right)(1+x)^{-5/2} = \frac{3}{8}(1+x)^{-5/2}$$

$$Q(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2$$
$$= 1 + \frac{1}{2}(1+0)^{-1/2}(x-0) + \frac{-\frac{1}{4}(1+0)^{-3/2}}{2}(x-0)^2$$
$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8}$$

Error: $|x| \leq .1 \Rightarrow -.1 \leq x \leq .1$

$$f'''(x) = \frac{3}{8\sqrt{(1+x)^5}} \qquad \text{at } -.1 \qquad \text{at } .1$$

$$\frac{3}{8\sqrt{.95}} \qquad \frac{3}{8\sqrt{1.15}}$$

$$= .48 \qquad = .29$$

$$\therefore M_3 = .48$$

$$|e_2(x)| \leq \frac{1}{6}(.48)|(0-.1)|$$
$$\leq \frac{1}{6}(.48)(.1)$$
$$\leq .00008$$

Assignment: Calculus p228(1,2,3,4)
AP Calculus p228(1,2,3,4)

J. Chapter 3 Test

Look at pages 230 - 235

Review Questions and
Miscellaneous Problems

IV. Chapter 4 - Integration

A. Indefinite Integrals - Section 4.1

There are 2 main types of integration (integration means - "to find the total").

1.) Definite Integral - In this integration, you find the exact curve that satisfies the differential equation. We will begin this in the next section.

2.) Indefinite Integrals - In this integration, we find a group of curves that satisfies the differential equation. This is what we will work on now.

$$\int f(x)\,dx = F(x) + C$$

Note: C is the constant of integration. It must be present after you integrate. This is why you have a "family" of curves for the indefinite integral.

Rules for Integration

1.) Constant Rule: $\int dx = x + C$
2.) Constant Multiple Rule: $\int a\,dx = a\int dx = ax + C$
3.) Sum Rule: $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
4.) Simple Power Rule: $\int x^n = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

Note: If $n=-1$, you must use the natural logarithmic function. We will get to this later.

These rules can be made general by letting u + v denote differentiable functions of x (as on the bottom of page 240).

1.) $\int \frac{du}{dx}\,dx = u(x) + C$

2.) $\int a\,u(x)\,dx = a\int u(x)\,dx$

3.) $\int [u(x) \pm v(x)]\,dx = \int u(x)\,dx + \int v(x)\,dx$

4.) $\int u^n \frac{du}{dx}\,dx = \frac{u^{n+1}}{n+1} + C \quad (n \neq -1)$

Examples: Find y, if given:

1.)
$$\begin{aligned}
\frac{dy}{dx} &= 4x^3 \\
dy &= 4x^3\,dx \\
\int dy &= \int 4x^3\,dx \\
y + C_1 &= 4\int x^3\,dx \\
y + C_1 &= 4 \cdot \frac{x^4}{4} + C_2 \\
y + C_1 &= x^4 + C_2 \\
y &= x^4 + C
\end{aligned}$$

2.)
$$\begin{aligned}
\frac{dy}{dx} &= 6 \\
dy &= 6\,dx \\
\int dy &= \int 6\,dx \\
y + C_1 &= 6\int dx \\
y + C_1 &= 6x + C_2 \\
y &= 6x + C
\end{aligned}$$

3.)
$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{x^4} \\
\frac{dy}{dx} &= x^{-4} \\
dy &= x^{-4}\,dx \\
\int dy &= \int x^{-4}\,dx \\
y + C_1 &= \frac{x^{-3}}{-3} + C_2 \\
y &= -\frac{1}{3}x^{-3} + C
\end{aligned}$$

4.) $\frac{dy}{dx} = 3x^2 - x + 4$

$\int dy = 3\int x^2 dx - \int x\,dx + 4\int dx$

$y + C_1 = 3\cdot\frac{x^3}{3} - \frac{x^2}{2} + 4x + C_2 + C_3 + C_4$

$y = x^3 - \frac{x^2}{2} + 4x + C$

5.) $\frac{dy}{dx} = \sqrt[5]{x}$

$dy = x^{1/5}\,dx$

$\int dy = \int x^{1/5}\,dx$

$y + C_1 = \frac{x^{6/5}}{6/5} + C_2$

$y = \frac{5}{6}x^{6/5} + C$

6.) $\frac{dy}{dx} = x^2(x^3-1)$

$dy = x^2(x^3-1)\,dx$

Let $u = x^3 - 1$

$du = 3x^2\,dx$ (Note: This says we need a "3" inside, to get that, we put a $\frac{1}{3}$ outside)

$n = 1$

$\int dy = \frac{1}{3}\int x^2(x^3-1)(3)\,dx$

$y + C_1 = \frac{1}{3}\,\frac{(x^3-1)^2}{2} + C_2$

$y = \frac{(x^3-1)^2}{6} + C$

7.) $\frac{dy}{dx} = \sqrt{xy}$ (Note: we must use "separation of variables")

$\frac{dy}{dx} = x^{1/2}y^{1/2}$

$dy = x^{1/2}y^{1/2}\,dx$

$y^{-1/2}dy = x^{1/2}dx$

$\int y^{-1/2}dy = \int x^{1/2}\,dx$

$\frac{y^{1/2}}{1/2} + C_1 = \frac{x^{3/2}}{3/2} + C_2$

$2y^{1/2} = \frac{2}{3}x^{3/2} + C$

$y^{1/2} = \frac{x^{3/2}}{3} + C$

$y = \left(\frac{x^{3/2}}{3} + C\right)^2$

Assignment: Calculus p243-244(1,2,4,6,7,12,17,22-26,30,
36,37)
APCalculus p243-244(3,5,8,10,16,22-26,30,
31,33,34,36,37)

B. Selecting a Value for the Constant of Integration - Section 4.2

When you are given initial (or Known) conditions, you can find "C" and there-by find the exact curve.

Examples:

1.) Given $f'(x) = 7 + x^{1/3}$ and $f(2) = 1/2$, find $f(x)$

$$\frac{dy}{dx} = 7 + x^{1/3}$$
$$\int dy = \int 7\,dx + \int x^{1/3}\,dx$$
$$y = 7x + \frac{3}{4}x^{4/3} + C$$

$$\therefore \quad \frac{1}{2} = 7(2) + \frac{3}{4}\sqrt[3]{16} + C$$
$$\frac{1}{2} = 14 + \left(\frac{3}{4}\right)(2)\sqrt[3]{2} + C$$
$$\frac{1}{2} = 14 + \frac{3}{2}\sqrt[3]{2} + C$$
$$C = \frac{1}{2} - 14 - \frac{3}{2}\sqrt[3]{2}$$
$$C = -15.38988$$

$$\therefore \quad f(x) = 7x + \frac{3}{4}x^{4/3} - 15.39$$

2.) Suppose the marginal cost $\left(\frac{dc}{dx}\right)$ for producing x units of a commodity is given by:

$$\frac{dC}{dx} = 32 - .04x$$

If it costs \$50.00 to make 1 unit, find the total cost of making 200 units

$$\frac{dC}{dx} = 32 - .04x$$
$$\int dC = \int 32\,dx - \int .04x\,dx$$
$$C = 32x - .02x^2 + C_1$$

when $x = 1$, $C = 50$

$$\therefore\ 50 = 32(1) - .02(1)^2 + C_1$$
$$C_1 = 50 - 32 + .02$$
$$C_1 = 18.02$$

$$\therefore\ C = 32x - .02x^2 + 18.02$$

at $x = 200$

$$C = 32(200) - .02(200)^2 + 18.02$$
$$C = 6400 - 800 + 18.02$$
$$C = 5618.02$$

\$5618.02

Do example #3 - This is a single stage rocket problem.

Do example #4 - This is a 2 stage rocket problem.

Note: These 2 examples are on the next 3 pages of my notes.

Example #3

5
+2
---
⑦

BROCK HAMULA
CALCULUS 6TH
5/10/88

A ROCKET IS LAUNCHED INTO SPACE. THE ROCKET HAS A CONSTANT NET ACCELERATION OF 90FT./SEC.^{2} FOR THE FIRST 20 SEC. ASSUME THE MASS OF THE ROCKET IS 800 SLUGS.

A.) FIND ACCELERATION DUE TO THRUST

B.) FIND THE FORCE ACTING ON THE ROCKET DUE TO GRAVITY

C.) FIND THE FORCE OF THRUST ACTING ON THE ROCKET

D.) FIND THE NET FORCE ACTING ON THE ROCKET

E.) FIND THE EQUATION FOR VELOCITY WHEN t=20 SEC V=2340 ft./sec

F.) FIND AN EQUATION FOR DISTANCE WITH RESPECT TO 't' WHEN t=20 ALTITUDE IS 35100 ft.

G.) AT THAT POINT t=20 THE ROCKET ENGINES SHUT DOWN. WHAT WILL THE SHIPS MAXIMUM ALTITUDE BE IN MILES?

*ACCELERATION DUE TO GRAVITY REMAINS CONSTANT

$A_{grav.} = -32 ft/sec^2$

A.) ACCELERATION DUE TO THRUST

$A_{net} = A_t + A_g$

$90 = A_t + (-32)$

$A_t = 90 + 32$

$A_t = \boxed{122 ft/sec^2}$

B.) FORCE DUE TO GRAVITY

$F_g = MA_g$

$F_g = 800(-32)$

$F_g = \boxed{-25600 \text{ lbs.}}$

C.) FORCE OF THRUST

$F_t = MA_t$

$F_t = 800(122)$

$F_t = \boxed{97600 \text{ lb.}}$

Example #3

D.) NET FORCE

1.)

$$F_n = MA_n$$

$$F_n = 800(90)$$

$$F_n = \boxed{72000 \text{ lbs.}}$$

2.)

$$F_n = F_t + F_g$$

$$F_n = 97600 + (-25600)$$

$$F_n = \boxed{72000 \text{ lbs.}}$$

E.) VELOCITY

$$\frac{dv}{dt} = 90 = A$$

$$dv = 90dt$$

$$\int dv = \quad 90dt$$

$$v = 90t + C$$

$$2340 = 90(20) + C$$

$$C = 2340 - 1800$$

$$C = 540$$

$$V = \boxed{90t + 540}$$

F.) DISTANCE

$$\frac{ds}{dt} = V = 90t + 540$$

$$\int ds = \int 90t + 540 \int dt$$

$$S = 45t^2 + 540t + C$$

$$35100 = 45(20)^2 + 540(20) + C$$

$$35100 = 18000 + 10800 + C$$

$$35100 = 28800 + C$$

$$C = 6300$$

$$S = \boxed{45t^2 + 540t + 6300}$$

G.)

$$S = S_o + V_o t + At^2$$

$$S = 35100 + 2340t + \tfrac{1}{2}At^2$$

$$\boxed{S = 35100 + 2340t + 16t^2}$$

$$\frac{ds}{dt} = 2340 - 32t$$

$$32t = 2340$$

$$t = 73.125 \text{ sec.}$$

$$S = 35100 + 2340(73.125) - 16(73.125)^2$$

$$206212.5 - 85556.25$$

$$120656.25 \times \frac{1 \text{ mile}}{5280 \text{ ft.}}$$

$$\boxed{22.85 \text{ MILES}}$$

⑤
+2
⑦

[illegible]
Calculus
per. 6

(76)

Example #4

Two-Stage Rocket Problem

A space ship is launched into space. At the burnout of its second stage, total time = 60 seconds, its V = 2010 ft/sec. Its distance (altitude) is 62550. Assume constant acceleration through the second burn at 40 ft/sec^2, also assume acceleration constant throughout the first stage at 20 ft/sec^2. The first stage burn ended at t = 25 seconds. Assume that no time is lost between the firing of stages.

a. find an equation for distance in terms of t for second stage
b. find an equation for distance in terms of t for first stage
c. at the end of the second burn, acceleration due to gravity is equal to -22 ft.sec^2, find maximum altitude ship will reach in miles.

a. $a = 40$
$dv/dt = 40$
$\int dv = \int 40\, dt$
$V = 40t + C$
$(2010) = 40(35) + C$
$C = 2010 - 1400$
$C = 610$
$\boxed{V = 40t + 610}$

$ds/dt = 40t + 610$
$\int ds = \int 40tdt + \int 610dt$
$S = 20t^2 + 610t + C$
$(62550) = 20(35)^2 + 610(35) + C$
$62550 = 20(1225) + 21350 + C$
$62550 + 24500 + 21350 + C$
$C = 62550 - 45850$
$C = 16700$
$\boxed{S = 20t^2 + 610t + 16700}$
$S = \frac{1}{2}at^2 + V_0t + S_0$

b. $a = 20$
$dv/dt = 20$
$\int dv = \int 20dt$
$V = 20t + C$
$(610) = 20(25) + C$
$610 = 500 + C$
$C = 610 - 500$
$C = 110$
$\boxed{V = 20t + 110}$

$ds/dt = 20t + 110$
$\int ds = \int 20tdt + \int 110dt$
$S = 10t^2 + 110t + C$
$(16700) = 10(25)^2 + 110(25) + C$
$16700 = 10(625) + 2750 + C$
$16700 = 6250 + 2750 + C$
$16700 = 9000 + C$
$C = 16700 - 9000$
$C = 7700$
$\boxed{S = 10t^2 + 110t + 7700}$

c. $S = S_0 + V_0t + \frac{1}{2}at^2$
$S = 62550 + 2010t + \frac{1}{2}(-22)t^2$
$\boxed{S = 62550 + 2010t - 11t^2}$

$ds/dt = 2010 - 22t$ $\qquad d^2s/dt^2 = -22$
$22t = 2010$ $\qquad -22 < 0 \Rightarrow$ max.
$t = 91.4$

$S = 62550 + 2010(91.4) - 11(91.4)^2$
$S = 62550 + 18714 - 91893.56$
$S = 154370.44$

$154370.44\, \cancel{ft} \times \frac{1 \text{ mile}}{5280\, \cancel{ft}} \cong \boxed{29.24 \text{ miles}}$

Assignment: Calculus p246-247 (1, 4, 7, 10, 13, 16, 19, 26, 28, 30)
AP Calculus p246-247 (3, 6, 8, 11, 14, 17, 20, 21, 24, 26, 27, 30)

C. The Substitution Method of Integration - Section 4.3

Integration by substitution can often be used to transform complicated integration problems into simpler ones.

Steps for integration by substitution:

1.) Make a choice for u, say $u = g(x)$
2.) Compute $\frac{du}{dx} = g'(x)$
3.) Make the substitution $u = g(x)$, $du = g'(x)\,dx$.
Note: The entire integral must be in terms of u; no x's should remain. If this is not the case, try a different choice of u.

4.) Evaluate the resulting integral.

5.) Replace u by $g(x)$, so the final answer is in terms of x.

Examples:

1.) $\int (x^2+1)^{50}\, 2x\,dx$

$u = x^2+1$
$du = 2x\,dx$
$n = 50$

$$\int u^{50}\,du$$
$$= \frac{u^{51}}{51} + C$$
$$= \frac{(x^2+1)^{51}}{51} + C$$

2.) $\int 3x^2 \sqrt{x^3+1}\, dx$

$= \int 3x^2 (x^3+1)^{1/2}\, dx$

$u = x^3+1$

$du = 3x^2\, dx$

$n = 1/2$

$\int u^{1/2}\, du$

$= \frac{u^{3/2}}{\frac{3}{2}} + C = \frac{2}{3} u^{3/2} + C$

$= \frac{2}{3}(x^3+1)^{3/2} + C$

3.) $\int \frac{dx}{(\frac{1}{3}x-8)^5} = \int (\frac{1}{3}x-8)^{-5} dx$

$u = \frac{1}{3}x-8$

$du = \frac{1}{3}dx$

$n = -5$

$3\int u^{-5} du = 3\left(\frac{u^{-4}}{-4}\right) + C$

$= 3\left(\frac{(1/3x-8)^{-4}}{-4}\right) + C$

$= -\frac{3}{4}(1/3x-8)^{-4} + C$

4.) $\int t^4 \sqrt[3]{3-5t^5}\, dt = \int t^4 (3-5t^5)^{1/3} dt$

$u = 3-5t^5$

$du = -25t^4\, dt$

$n = 1/3$

$-\frac{1}{25}\int u^{1/3} du$

$= -\frac{1}{25}\frac{u^{\frac{4}{3}}}{\frac{4}{3}} + C = -\frac{3}{100}u^{4/3} + C$

$= -\frac{3}{100}(3-5t^5)^{4/3} + C$

✓ $\int (x^3+1)^4 (3x^2)\,dx$

✓ $\int (x^2+2x+7)^5 (x+1)\,dx$

$\int x(x^2+3)^{1/3}\,dx$

$\int (x^3-3x)^{2/5}(x^2-1)\,dx$

$\int (x^2-2x+3)^{4/3}(x-1)\,dx$

5.) $\int x^2\sqrt{x-1}\,dx = \int x^2(x-1)^{1/2}\,dx$

Let the student try to think this one out.

$u = x-1$
$du = dx$

$\Rightarrow \quad x = u+1 \Rightarrow x^2 = (u+1)^2 = u^2+2u+1$

$\therefore \quad \int x^2(x-1)^{1/2}\,dx = \int (u^2+2u+1)\,u^{1/2}\,du$

$= \int (u^{5/2} + 2u^{3/2} + u^{1/2})\,du$

$= \frac{2}{7}u^{7/2} + \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C$

$= \frac{2}{7}(x-1)^{7/2} + \frac{4}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C$

Assignment: Calculus p252-253 (1, 2, 5, 6, 8, 9, 11, 12, 13, 14, 15,
16, 19, 21, 24)

AP Calculus p252-253 (3, 4, 7, 10, 17, 18, 20,
23, 25, 28, 32, 33, 35)

D. Integrals of Trigonometric Functions - Section 4.4

You must Know these 6 formulas and also be able to use the Trig identities.

Formulas:

1.) $\int \cos u \, du = \sin u + C$
2.) $\int \sin u \, du = -\cos u + C$
3.) $\int \sec^2 u \, du = \tan u + C$
4.) $\int \csc^2 u \, du = -\cot u + C$
5.) $\int \sec u \tan u \, du = \sec u + C$
6.) $\int \csc u \cot u \, du = -\csc u + C$

Trig identities that may be helpful:

1.) $\sin^2\theta + \cos^2\theta = 1$ (and all the various forms of this identity)

2.) $\cos^2\theta = \frac{1+\cos 2\theta}{2}$

3.) $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

Here is a trick to help integrate powers of Trig functions of this form:

$\int \cos^m\theta \sin^n\theta \, d\theta$, where m+n are nonnegative integers.

1.) If n is odd, pair one $\sin\theta$ with $d\theta$ to form $\sin\theta \, d\theta = -d(\cos\theta)$, and use the identity $\sin^2\theta = 1-\cos^2\theta$ together with the substitution $u = \cos\theta$. The integrand will be a polynomial in u.

2.) If m is odd, pair one $\cos\theta$ with $d\theta$ to form $\cos\theta \, d\theta = d(\sin\theta)$, and use the identity $\cos^2\theta = 1-\sin^2\theta$ together with the substitution $u = \sin\theta$. The integrand will be a polynomial in u.

Examples: (look at the 12 examples in the book pps 254-258)

1.) $\int \sin 4x\, dx \qquad u = 4x$

$du = 4\,dx \Rightarrow dx = \frac{1}{4}\,du$

$$\frac{1}{4}\int \sin u\, du$$
$$= -\frac{1}{4}\cos u + C = -\frac{1}{4}\cos 4x + C$$

2.) $\int \sin^5\theta\, d\theta$

$$= \int \sin^4\theta \sin\theta\, d\theta$$
$$= \int (1-\cos^2\theta)^2 \sin\theta\, d\theta$$
$$= \int (1-2\cos^2\theta + \cos^4\theta)\sin\theta\, d\theta$$
$$= \int \sin\theta\, d\theta - 2\int \cos^2\theta \sin\theta\, d\theta + \int \cos^4\theta \sin\theta\, d\theta$$

$u = \cos\theta$

$du = -\sin\theta\, d\theta$

$$= -\cos\theta + \frac{2}{3}\cos^3\theta - \frac{1}{5}\cos^5\theta + C$$

3.) $\int \cos^3\theta \sin^4\theta\, d\theta$

$$= \int \cos^2\theta \sin^4\theta \cos\theta\, d\theta$$
$$= \int (1-\sin^2\theta)\sin^4\theta \cos\theta\, d\theta$$
$$= \int \sin^4\theta \cos\theta\, d\theta - \int \sin^6\theta \cos\theta\, d\theta$$

$u = \sin\theta$

$du = \cos\theta$

$$= \frac{1}{5}\sin^5\theta - \frac{1}{7}\sin^7\theta + C$$

4.) $\int \cos^m 2\theta \sin^3 2\theta\, d\theta$

$$= \int \cos^m 2\theta \sin^2 2\theta \sin 2\theta\, d\theta$$
$$= \int \cos^m 2\theta (1-\cos^2 2\theta)\sin 2\theta\, d\theta$$
$$= \int \cos^m 2\theta \sin 2\theta\, d\theta - \int \cos^{m+2} 2\theta \sin 2\theta\, d\theta$$

$u = \cos 2\theta$

$du = -2\sin 2\theta\, d\theta$

$$= -\frac{1}{2(m+1)}\cos^{m+1} 2\theta + \frac{1}{2(m+3)}\cos^{m+3} 2\theta + C$$

Assignment: Calculus p258-259 (1, 3, 5, 7, 9, 10, 12, 13) p258-259 (15, 16, 20, 23, 31, 36, 40, 43, 47, 53)

AP Calculus p258-259 (2, 4, 6, 8, 14, 17, 21, 22, 27, 34, 43, 47, 51, 54)

E. Definite Integrals: Area under a curve - Section 4.5

This section will show you how to find the area under a curve. The method used was the first method used.

The basic idea is to divide the area into small rectangles and find the area of these rectangles and add them together.

The new concept that is used is: <u>Summation</u>

$$S_n = f(c_1)\Delta x + f(c_2)\Delta x + \cdots + f(c_n)\Delta x$$

which is written as

$$S_n = \sum_{k=1}^{n} f(c_k)\Delta x$$

Example: Find the value of the sum

$$\sum_{k=1}^{3} k = 1+2+3 = 6$$

$$\begin{aligned}\sum_{k=0}^{4}(k+1)^2 &= (0+1)^2+(1+1)^2+(2+1)^2+(3+1)^2+(4+1)^2\\ &= 1+4+9+16+25\\ &= 55\end{aligned}$$

The area under a curve is:

$$A = \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x$$

Put Table 4.1 on page 269 into your notebook's.

Example: Given $\int_0^4 f(x)\,dx = 3$, $\int_4^6 f(x)\,dx = 7$

Find: $\int_0^6 f(x)\,dx$ and $\int_6^4 f(x)\,dx$

a.) $\int_0^6 f(x)\,dx = \int_0^4 f(x)\,dx + \int_4^6 f(x)\,dx = 3+7 = 10$

b.) $\int_6^4 f(x)\,dx = -\int_4^6 f(x)\,dx = -7$

Assignment: Calculus p270-271 (1, 2, 6, 10, 11, 14, 16)

APCalculus p270-271 (2, 3, 7, 10, 13, 15, 16, 17)

F. The Fundamental Theorems of Integral Calculus - Section 4.7

There are 2 Fundamental Theorems of Integral Calculus:

1.) The first one states that:

If $F(x) = \int_0^x f(t)\,dt$, then

$$\frac{dF}{dx} = \frac{d}{dx}\int_0^x f(t)\,dt = f(x)$$

2.) The second one is the one that we will concentrate on. It allows us to find the area "under" a curve. It states:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

This has many uses. For example, $W = F \cdot d = \int F\,ds$.

Examples: Evaluate

1.) $\int_1^2 x\,dx$

$$= \frac{x^2}{2}\Big|_1^2 = (\tfrac{1}{2})(2)^2 - (\tfrac{1}{2})(1)^2 = 2 - \tfrac{1}{2} = \frac{3}{2}$$

2.) $\int_0^3 (x^3-4x+1)\,dx$

$= \left(\frac{x^4}{4} - 2x^2 + x\right)\Big|_0^3 = \left(\frac{81}{4} - 18 + 3\right) - (0-0+0) = \frac{21}{4}$

3.) Find the area bounded by the x-axis, the curve $y=x^3$, and the vertical lines $x=2$ and $x=3$.

$y = x^3$, $x=2$, $x=3$

$A = \int_2^3 x^3\,dx$

$= \frac{x^4}{4}\Big|_2^3 = \frac{81}{4} - \frac{16}{4} = \frac{65}{4}$

4) Find the area under the curve $y = \cos x$, over the interval $[0, \frac{\pi}{2}]$

$A = \int_0^{\frac{\pi}{2}} \cos x\,dx$

$= \sin x\Big|_0^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1$

Assignment: Calculus p285-286 (1, 4, 7, 9, 15, 21, 23, 26, 32)

AP Calculus p285-286 (3, 6, 8, 13, 15, 18, 24, 27, 29, 38)

G. Substitution in Definite Integrals - Section 4.8

In this section we will discuss 2 methods of evaluating definite integrals for which a substitution is needed.

Method 1: Evaluate the indefinite integral ($\int h(x)\,dx$) and then use the relationship $\int_a^b h(x)\,dx = [\int h(x)\,dx]_a^b$ to evaluate the definite integral.

Method 2: Express the integral in the form $\int_a^b h(x)\,dx = \int_a^b f(g(x))\,g'(x)\,dx$ and then make the substitution $u = g(x)$ and $du = g'(x)\,dx$. You must then change the limits using:
$a = g(a)$, $b = g(b)$

Examples: Evaluate

1.) Method 1: $\int_0^2 2x(x^2+1)^3\,dx = \left.\frac{(x^2+1)^4}{4}\right|_0^2$

$\frac{625}{4} - \frac{1}{4} = \frac{624}{4} = 156$

Method 2: $\int_0^2 2x(x^2+1)^3\,dx$ $\quad u = x^2+1$, $du = 2x\,dx$, $n=3$

$a = (0^2+1) = 1$

$b = (2^2+1) = 5$

$= \left.\frac{u^4}{4}\right|_1^5$

$= \frac{625}{4} - \frac{1}{4} = \frac{624}{4} = 156$

2) Method 1: $\int_0^{\frac{\pi}{4}} \cos(\pi - x)\,dx = -\sin(\pi - x)\Big|_0^{\frac{\pi}{4}}$

$= -\sin(\frac{3\pi}{4}) - (-\sin(\pi)) = -\frac{1}{\sqrt{2}} + 0 = -\frac{1}{\sqrt{2}}$

Method 2: $\int_0^{\frac{\pi}{4}} \cos(\pi - x)\,dx$ $\quad u = \pi - x$, $du = -dx$

$a = \pi - 0 = \pi$, $b = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$= \int_{\pi}^{\frac{3\pi}{4}} \cos u(-du)$

$= -\int_{\pi}^{\frac{3\pi}{4}} \cos u\,du = -\sin u\Big|_{\pi}^{\frac{3\pi}{4}}$

$= -(\sin(\frac{3\pi}{4}) - \sin(\pi)) = -(\frac{1}{\sqrt{2}} - 0) = -\frac{1}{\sqrt{2}}$

3.) Method 1: $\int_0^{\frac{\pi}{8}} \sin^5 2x \cos 2x \, dx$

$= \left(\frac{1}{2}\right)\frac{\sin^6 2x}{6}\Big|_0^{\frac{\pi}{8}} = \frac{\sin^6 2x}{12}\Big|_0^{\frac{\pi}{8}} = \frac{\sin^6 \frac{\pi}{4}}{12} - \frac{\sin^6 0}{12}$

$= .0104 - 0 = .0104$

Method 2: $\int_0^{\frac{\pi}{8}} \sin^5 2x \cos 2x \, dx$

$u = \sin 2x$, $du = 2\cos 2x \, dx$

$n = 5$

$a = \sin 2(0) = 0$

$b = \sin 2(\frac{\pi}{8}) = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} = .7071$

$\frac{1}{2}\int_0^{.7071} u^5 \, du$

$= \left(\frac{1}{2}\right)\frac{u^6}{6}\Big|_0^{.7071} = \frac{u^6}{12}\Big|_0^{.7071}$

$= .0104 - 0 = .0104$

Assignment: Calculus p289 (1, 4, 8, 11, 15)
AP Calculus p289 (2, 5, 9, 10, 12, 16, 19)

H. Rules for Approximating Definite Integrals - Section 4.9

In this section we will talk about 2 ways of approximating definite integrals. There a 3 good reasons to learn this:

1.) The integrand $f(x)$ may not have an elementary antiderivative
2.) The antiderivative may be tedious to compute
3.) Some of the values of the integrand may not be known, such as: temperature may be recorded hourly.

Method #1: Trapezoidal Method - this method uses trapezoids instead of rectangles.

$\int_a^b f(x) \, dx$, the approximation is:

$$T = \frac{h}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right)$$

in n subintervals, where length $h = \frac{(b-a)}{n}$

The error in the trapezoidal method is:

$$|E_T| \le \frac{b-a}{12} h^2 M,$$ where M is the max. value of the second derivative (f'').

Examples: Estimate $\int_0^1 \frac{dx}{1+x^2}$ with $n=4$

In this case, $a=0, b=1, n=4 \Rightarrow h = \frac{(1-0)}{4} = \frac{1}{4}$

$$T = \frac{h}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right)$$

x_i	$f(x_i)$	Number
0	$\frac{1}{1+0^2}$	1.0000
$\frac{1}{4}$	$\frac{1}{1+(\frac{1}{4})^2}$	1.8824
$\frac{2}{4}$	$\frac{1}{1+(\frac{2}{4})^2}$	1.6000
$\frac{3}{4}$	$\frac{1}{1+(\frac{3}{4})^2}$	1.2800
1	$\frac{1}{1+1^2}$	.5000

$$\therefore T = \frac{\frac{1}{4}}{2}\left(1 + 1.8824 + 1.6 + 1.28 + .5\right) = \frac{1}{8}(6.2624)$$

$$T = .7828$$

Estimate $\int_0^1 x^2\,dx$, with $n=3$ and find the error

$a=0, b=1, n=4 \Rightarrow h = \frac{(1-0)}{3} = \frac{1}{3}$

$$T = \frac{h}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right)$$

x_i	$f(x_i)$	Number
0	0^2	0
$\frac{1}{3}$	$(\frac{1}{3})^2$	$\frac{1}{9} = .1111$
$\frac{2}{3}$	$(\frac{2}{3})^2$	$\frac{4}{9} = .4444$
1	1^2	1

$$\therefore T = \frac{\frac{1}{3}}{2}\left(0 + .2222 + .8888 + 1\right) = \frac{1}{6}(2.111)$$

$$T = .35183$$

$$|E_T| \leq \frac{b-a}{12} h^2 M$$

$M = 2$

$y = x^2$

$\frac{dy}{dx} = 2x$

$\frac{d^2y}{dx^2} = 2$

$$\therefore |E_T| \leq \frac{1-0}{12}\left(\frac{1}{3}\right)^2 (2)$$
$$\leq \frac{1}{12}\left(\frac{1}{9}\right)(2)$$
$$\leq \frac{1}{54}$$
$$\leq .01852$$

The exact value is: $\int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3} - 0 = \frac{1}{3} = .333\overline{3}$

Method#2: Simpson Rule - this method uses the parabola.

To approximate $\int_a^b f(x)\,dx$, we use

$$S = \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n\right)$$

n must be even, and $h = \frac{(b-a)}{n}$

The error in the Simpson Rule is less than the trapezoidal rule, and is given by:

$|E_S| \leq \frac{b-a}{180} h^4 M$, where M is the max value of the 4th derivative of $f(x)$

Estimate $\int_0^1 \frac{dx}{1+x^2}$ with $n=4$

$a=0$, $b=1$, $n=4 \Rightarrow h = \frac{1-0}{4} = \frac{1}{4}$

$$S = \frac{h}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right)$$

x_i	$f(x_i)$	Number
0	$\frac{1}{1+0^2}$	1.0000
$\frac{1}{4}$	$\frac{1}{1+(\frac{1}{4})^2}$	3.7647
$\frac{2}{4}$	$\frac{1}{1+(\frac{2}{4})^2}$	1.6000
$\frac{3}{4}$	$\frac{1}{1+(\frac{3}{4})^2}$	2.5600
1	$\frac{1}{1+1^2}$	.5000

$$S = \frac{\frac{1}{4}}{3}(1.0000 + 3.7647 + 1.6000 + 2.5600 + .5000)$$
$$= \frac{1}{12}(9.4247)$$
$$= .78539$$

Estimate $\int_0^1 x^2\,dx$ with $n=4$ and find the error

$a=0, b=1, n=4 \Rightarrow h = \frac{1-0}{4} = \frac{1}{4}$

$$S = \frac{h}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4))$$

x_i	$f(x_i)$	Number
0	0^2	0
$\frac{1}{4}$	$(\frac{1}{4})^2$	$(\frac{1}{16})(4) = \frac{1}{4}$
$\frac{2}{4}$	$(\frac{2}{4})^2$	$(\frac{4}{16})2 = \frac{4}{8} = \frac{1}{2}$
$\frac{3}{4}$	$(\frac{3}{4})^2$	$(\frac{9}{16})(4) = \frac{9}{4}$
1	1^2	1

$$S = \frac{\frac{1}{4}}{3}(0 + \frac{1}{4} + \frac{1}{2} + \frac{9}{4} + 1) = \frac{1}{12}(\frac{16}{4}) = \frac{1}{3} = .333\overline{3}$$

$\cancel{y} = x^2$ $\quad f'' = 2$ $\quad f^{IV} = 0$

$\cancel{\frac{dy}{dx}} = 2x$ $\quad f''' = 0$

$$|E_s| \leq \frac{1-0}{180}\left(\frac{1}{4}\right)^4(0)$$
$$\leq 0 \Rightarrow \text{no error}$$

Note: If the polynomial is less than 4th degree, there will be no error using Simpson Rule.

For $\int_0^1 x^2\,dx$, estimate the min. number of subdivisions needed to approximate the integral with an error of less than 10^{-5} by a.) trapezoidal, and b.) Simpson's rule.

a.) $f = x^2$, $f' = 2x$, $f'' = 2$

$$\therefore E_T \leq \frac{b-a}{12} h^2 M$$
$$\leq \frac{1-0}{12}\left(\frac{1}{n}\right)^2(2) = \frac{1}{12}\left(\frac{1}{n}\right)^2(2) = \frac{1}{6n^2}$$

$$\therefore 10^{-5} = \frac{1}{6n^2}$$
$$6n^2 = 100000$$
$$n^2 = \frac{100000}{6} = 16666.7$$
$$n = 129.09$$

This says that n must be greater than 129 or, at least 130 subdivisions.

b.) $f = x^2$, $f' = 2x$, $f'' = 2$, $f''' = 0$, $f^{IV} = 0$

This says that any even number of n will give the exact answer, or $n \geq 2$.

Assignment: Calculus p299 (1, 4, 10, 11)
AP Calculus p299 (2, 3, 5, 9, 12, 13)

I. Chapter 4 Test

Look at pages 302-304

V. Chapter 5 - Applications of Definite Integrals

A. Change in Position and Distance Traveled - Section 5.1

As seen before, $v = \frac{ds}{dt}$, which implies:

$$s(t) = \int v(t)\,dt = F(t) + C$$

From this, we can calculate 2 items:

1.) displacement - the net change in position

$$\int_a^b v(t)\,dt = s(t)\Big|_a^b = S(b) - S(a)$$

2.) total distance traveled

$$\int_a^b |v(t)|\,dt$$

Note: The absolute value of the velocity is called the body's SPEED.

Example: ① Find the position function of a particle moving with velocity $v(t) = \cos \pi t$, assuming the particle is at $s = 4$ when $t = 0$ ($S(0) = 4$).

$$s(t) = \int v(t)\,dt = \int \cos \pi t\,dt = \frac{1}{\pi} \sin \pi t + C$$

$$4 = S(0) = \frac{1}{\pi} \sin 0 + C$$

$$C = 4$$

$$\therefore\ S(t) = \frac{1}{\pi} \sin \pi t + 4$$

Example: ② A particle moves on a coordinate line so that its velocity at time t is $v(t) = t^2 - 2t$ m/s. Find

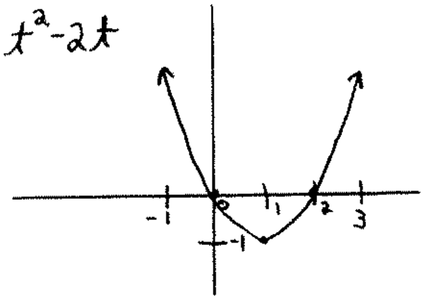
a.) Graph $v(t)$, to find where it is positive and negative.

b.) the total distance traveled for the given interval. $0 \leq t \leq 3$

C.) the net change in the body's position. $0 \le t \le 3$

a.) Graph: $v(t) = t^2 - 2t$

t	$v(t)$
1	-1
0	0
2	0
3	3



$y = t^2 - 2t$

$\frac{dy}{dt} = 2t - 2$

$0 = 2t - 2$

$t = 1$

$\frac{d^2y}{dt^2} = 2 > 0 \Rightarrow$ abs. min.

b.) On the interval $0 \le t \le 3$, $v(t)$ is positive on $2 \le t \le 3$ and negative for $0 \le t \le 2$

Total distance $= \int_0^3 |v(t)|\,dt = \int_0^2 -v(t)\,dt + \int_2^3 v(t)\,dt$

$$= \int_0^2 -(t^2 - 2t)\,dt + \int_2^3 (t^2 - 2t)\,dt$$

$$= -\left(\frac{t^3}{3} - t^2\right)\Big|_0^2 + \left(\frac{t^3}{3} - t^2\right)\Big|_2^3$$

$$= \left[-\left(\frac{8}{3} - \frac{12}{3}\right) + \left(\frac{0}{3} - 0\right)\right] + \left(\frac{27}{3} - \frac{27}{3}\right) - \left(\frac{8}{3} - \frac{12}{3}\right)$$

$= \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$ meters

C.) displacement $= \int_0^3 v(t)\,dt \quad 0 \le t \le 3$

$$= \int_0^3 (t^2 - 2t)\,dt = \left(\frac{t^3}{3} - t^2\right)\Big|_0^3$$

$$= \left(\frac{27}{3} - 9\right) - \left(\frac{0}{3} - 0\right) = 0$$

Assignment: Calculus p308 (2-6), p308 (9, 12, 14)
AP Calculus p308 (2-6), p308 (10, 11, 13, 16)

B. Areas Between Curves - Section 5.2

Systematic Procedure to set up the formula:

1.) Sketch the region and then draw a vertical line segment connecting the top and bottom boundaries.

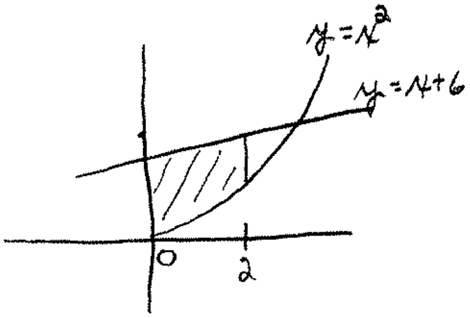
2.) The top segment will be $f_1(x)$, the bottom segment will be $f_2(x)$, and the length of the line segment will be $f_1(x) - f_2(x)$. This is the integrand.

3.) To determine the limits of integration, imagine moving the line segment left and then right. The left most position is the $x=a$ value, and the rightmost position is the $x=b$ value.

The formula for area is:

$$\text{Area} = \int_a^b (f_1(x) - f_2(x))\,dx$$

Example #1: Find the area of the region bounded above by $y=x+6$, bounded below by $y=x^2$, and bounded on the sides by the lines $x=0$ and $x=2$



$$A = \int_a^b (f_1(x)-f_2(x))\,dx$$

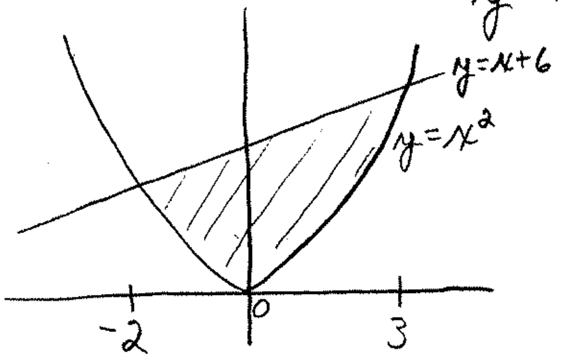
$f_1(x) = x+6,\ f_2(x) = x^2$

$a=0,\ b=2$

$$A = \int_0^2 [(x+6) - x^2]\,dx$$

$$= \frac{x^2}{2} + 6x - \frac{x^3}{3}\Big|_0^2 = \frac{34}{3} - 0 = \frac{34}{3}$$

Example #2: Find the area of the region enclosed between the curves $y = x + 6$ and $y = x^2$.



Find where the curves intersect.

$y = x+6$, $y = x^2$

$\Rightarrow x^2 = x+6$

$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$x = 3, -2$

$$\text{Area} = \int_{-2}^{3} [(x+6) - x^2]\,dx$$

$$= \frac{x^2}{2} + 6x - \frac{x^3}{3} \Big|_{-2}^{3}$$

$$= \frac{27}{2} - \left(-\frac{22}{3}\right) = \boxed{\frac{125}{6}}$$

Assignment: Calculus p312 (1, 3, 4, 6, 9, 19, 26, 28, 29)

AP Calculus p 312 (2, 5, 7, 8, 15, 19, 23, 26, 28, 29)

p312 (1, 6, 10, 13)

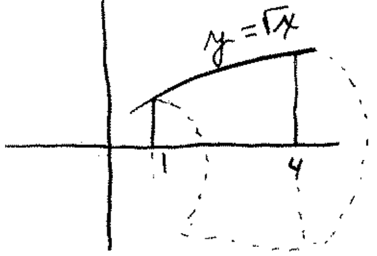
C. Calculating Volumes by Slicing. Volumes of Revolution - Section 5.3

We will use definite integrals to find volumes of 3-dimensional solids.

1.) Volumes by disk method (Perpendicular to the X-axis):

$$V = \int_a^b \pi [f(x)]^2 \, dx$$

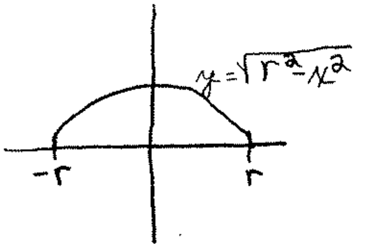
Example #1: Find the volume of the solid obtained when the region under the curve $y = \sqrt{x}$ over the interval $[1, 4]$ is revolved about the X-axis.



$$V = \int_a^b \pi [f(x)]^2 dx$$
$$= \int_1^4 \pi (\sqrt{x})^2 dx$$
$$= \int_1^4 \pi x \, dx = \frac{\pi x^2}{2} \Big|_1^4$$
$$= 8\pi - \frac{\pi}{2} = \frac{15\pi}{2}$$

Example #2: Derive the formula for the volume of a sphere of radius r.

Note: You can make a sphere of radius r by revolving the upper half of the circle: $x^2 + y^2 = r^2$ about the X-axis. Since the upper half only of the circle is graphed:



$$y = f(x) = \sqrt{r^2 - x^2}$$

$$V = \int_a^b \pi [f(x)]^2 dx$$

$$= \int_{-r}^{r} \pi \left(\sqrt{r^2-x^2}\right)^2 dx$$

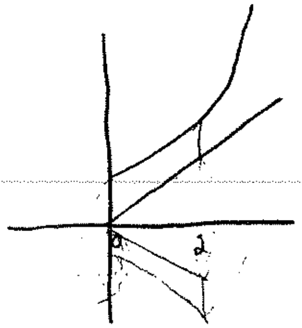
$$= \int_{-r}^{r} \pi (r^2-x^2) dx$$

$$= \pi \left(r^2 x - \frac{x^3}{3}\right)\Big|_{-r}^{r} = \frac{4}{3}\pi r^3$$

2.) <u>Volumes by Washer Method</u> (Perpendicular to the X-axis):

$$V = \int_a^b \pi \left([f(x)]^2 - [g(x)]^2 \right) dx$$

Example: Find the volume of the solid generated when the region between the graphs of: $f(x) = \frac{1}{2} + x^2$ and $g(x) = x$ over the interval $[0, 2]$ is revolved about the X-axis.



$$V = \int_a^b \pi \left([f(x)]^2 - [g(x)]^2 \right) dx$$

$$= \int_0^2 \pi \left([\tfrac{1}{2} + x^2]^2 - [x]^2 \right) dx$$

$$= \int_0^2 \pi \left(\tfrac{1}{4} + x^2 + x^4 - x^2 \right) dx$$

$$= \int_0^2 \pi \left(\tfrac{1}{4} + x^4 \right) dx$$

$$= \pi \left(\frac{x}{4} + \frac{x^5}{5} \right)\Big|_0^2$$

$$= \frac{69\pi}{10}$$

Chapter 5.3 - Volumes by Slicing

This section covers 2 methods of calculating volume:

1.) The Disk Method: Integrate from a to b of pi times f(x) squared.
2.) The Washer Method: Integrate from a to b of pi times [f(x) squared - g(x) squared].

Let function f be nonnegative and continuous on [a,b], and let R be the region bounded above by the graph of f, below by the x-axis, and on the sides by the lines x=a and x=b. When this region is revolved about the x-axis, it generates a solid having a circular cross section. Since the cross section at x has a radius f(x), the cross-sectional area is A(x) = pi[f(x)] squared. Because the cross sections are circular (or disk shaped), this formula is called the Disk Method.
Problem: Find the volume of the solid obtained when the region under the curve y = x^(½) over the interval [1,4] is revolved about the x-axis. Answer: 15pi/2

Now lets consider more general solids of revolution. For this we need 2 nonnegative continuous functions such that g(x) <= f(x) over [a,b], and R is the region enclosed between the graphs of the functions and the lines x=a and x=b. When this region is revolved about the x-axis, it generates a solid having annular or washer-shaped cross sections. Since the cross section at x has an inner radius of g(x) and an outer radius of f(x), its area is A(x) = pi([f(x)]squared - [g(x)]squared).
Problem: Find the volume of the solid generated when the region between the graphs of f(x) = (½) + x^2 and g(x) = x over the interval [0,2] is revolved about the x-axis. Answer: 69pi/10

The rules for which one to use is simple, and why your book does not state this is beyond me. If the regions cross section is disk shaped (circular) then you will use the Disk Method. Generally this is when the curve is bounded by either the x or y axis. If the regions cross section is washer shaped (has a hole in the middle), you will use the Washer Method. Generally this is when you are trying to find the volume between two curves, not bounded by the x or y axis.

The analogs for regions revolved about the y-axis is that you must solve for x and not y, and integrate with respect to y and not x. For example: Find the volume of the solid generated when the region enclosed by y = x^(½), y = 2, and x = 0 is revolved about the y-axis. The cross sections taken perpendicular to the y-axis are disks (or circular), therefore you must use the Disk Method. But first you must rewrite the equation y = x^(½) as x = y^2. Finish this problem. Answer: 32pi/5

I hope this clears up the slight confusion with these two. The real key is that you need to see the graphs of the functions, so you can tell what type of cross section you are dealing with.

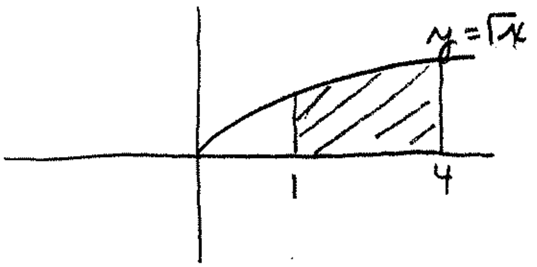
Assignment: Calculus p318 (1, 2, 5, 6, 11, 20)

AP Calculus p318 (3, 4, 8, 9, 13, 16, 20)

D. Volumes Modeled with Washers and Cylindrical Shells - Section 5.4

$$V = \int_a^b 2\pi x f(x)\, dx$$

Example 1: Use cylindrical shells to find the volume of the solid generated when the region enclosed between $y = \sqrt{x}$, $x = 1$, $x = 4$, and the x-axis is revolved about the y-axis.



$f(x) = \sqrt{x}$, $a = 1$, $b = 4$

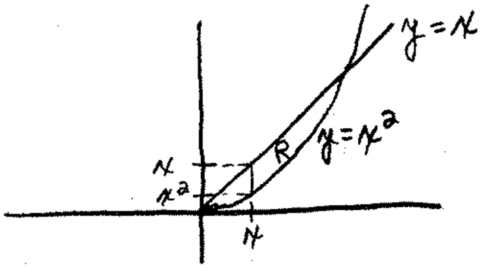
$$V = \int_1^4 2\pi x (\sqrt{x})\, dx$$

$$= \int_1^4 2\pi x^{3/2}\, dx$$

$$= 2\pi \cdot \frac{2}{5} x^{5/2} \Big|_1^4$$

$$= \frac{4\pi}{5}(32-1) = \frac{124\pi}{5}$$

Example 2: Use cylindrical shells to find the volume of the solid generated when the region R in the first quadrant enclosed between $y = x$ and $y = x^2$ is revolved about the y-axis.



Solution:

At each x in $[0,1]$, the cross section of R parallel to the y-axis generates a cylindrical surface of height $x-x^2$ and radius x. Since the area of this surface is

$$2\pi x(x-x^2)$$

the volume of the solid is:

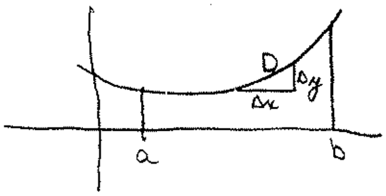
$$V = \int_0^1 2\pi x(x-x^2)\,dx = 2\pi\int_0^1 (x^2-x^3)\,dx$$
$$= 2\pi\left(\frac{x^3}{3}-\frac{x^4}{4}\right)\Big|_0^1 = 2\pi\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{\pi}{6}$$
$$V = \frac{\pi}{6}$$

Assignment: Calculus p328 (1, 4, 5, 6, 13, 15, 20, 24)

AP Calculus p328 (2, 3, 7, 8, 13, 14, 16, 21, 24)

E. Lengths of Plane Curves - Section 5.5

In the section we will learn to find the length of Plane curves. To find the length of the curve, we use triangles to find the lengths.



$$D = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

What we need to do is sum these all up:

$$\sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

Knowing $\frac{\Delta y}{\Delta x} = f'(x)$

$$\therefore \quad \sum_{k=1}^{n} \sqrt{1 + f_k'(c_k)^2}$$

$$\Rightarrow \quad L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Example: Find the arc length of $y = x^{3/2}$ from $x = 1$ to $x = 2$

$$f'(x) = \frac{dy}{dx} = \frac{3}{2} x^{1/2}$$

$$L = \int_1^2 \sqrt{1 + (\tfrac{3}{2} x^{1/2})^2}\, dx$$

$$= \int_1^2 \sqrt{1 + \tfrac{9}{4} x}\, dx \qquad \begin{aligned} n &= \tfrac{1}{2} \\ u &= 1 + \tfrac{9}{4} x \\ du &= \tfrac{9}{4} dx \end{aligned}$$

$$= \frac{4}{9} \int_1^2 \frac{9}{4} (1 + \tfrac{9}{4} x)^{1/2} dx$$

$$= \left(\frac{4}{9}\right)\left(1 + \tfrac{9}{4} x\right)^{3/2} \left(\frac{2}{3}\right) \Big|_1^2 = \frac{8}{27}\left(1 + \tfrac{9}{4} x\right)^{3/2} \Big|_1^2$$

$$= \left[\frac{8}{27}(1 + \tfrac{9}{4}(2))^{3/2} - \frac{8}{27}(1 + \tfrac{9}{4}(1))^{3/2}\right] = (3.8218 - 1.736)$$

$$= 2.0858$$

Assignment: page 333 (1, 2, 3)

F. The Area of a Surface of Revolution - Section 5.6

In this section we will learn how to find the Surface Area when a curve is revolved.

$$S = \int_a^b 2\pi y \sqrt{1+(f'(x))^2}\,dx$$

Example#1: Find the Surface Area by revolving the curve

$$y = \sqrt{1-x^2}\ ,\quad 0 \le x \le \tfrac{1}{2}$$

$$f(x) = (1-x^2)^{1/2}$$

$$f'(x) = \tfrac{1}{2}(1-x^2)^{-1/2}(-2x) = -\frac{x}{\sqrt{1-x^2}}$$

$$S = \int_0^{1/2} 2\pi y \sqrt{1+(f'(x))^2}\,dx$$

$$S = \int_0^{1/2} 2\pi\left(\sqrt{1-x^2}\right)\left(\sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\right)dx$$

$$= \int_0^{1/2} 2\pi\left(\sqrt{1-x^2}\right)\left(\sqrt{\frac{1-x^2+x^2}{1-x^2}}\right)dx$$

$$= \int_0^{1/2} 2\pi\,dx$$

$$= 2\pi x\Big|_0^{1/2} = \pi - 0 = \boxed{\pi}$$

Assignment: page 340 (1,2,3)

G. The Average Value of a Function - Section 5.7

In this section we will learn how to find the average value of a function on a given interval.

$$y_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Example: Find the average value of the function $y = \sqrt{x}$ from $x=0$ to $x=4$

$$\begin{aligned} y_{ave} &= \frac{1}{b-a}\int_a^b f(x)\,dx \\ &= \frac{1}{4-0}\int_0^4 x^{1/2}\,dx \\ &= \frac{1}{4}\,\frac{x^{3/2}}{3/2}\Big|_0^4 = \left(\frac{1}{4}\right)\left(\frac{2}{3}\right)x^{3/2}\Big|_0^4 \\ &= \frac{1}{6}\,x^{3/2}\Big|_0^4 \\ &= \frac{1}{6}(4)^{3/2} - \frac{1}{6}(0)^{3/2} = \frac{1}{6}(8) = \boxed{\frac{4}{3}} \end{aligned}$$

Assignment: page 344 (1,2,3)

VI. Transcendental Functions

D. The Natural logarithm and its Derivative - Section 6.4

The <u>natural logarithm</u> function is defined by the formula:

$$\boxed{\ln x = \int_1^x \frac{1}{t}\,dt,\ x > 0}$$

Using the chain rule and the Second Fundamental Theorem of Calculus, we find that the general formula is:

$$\frac{d(\ln u)}{dx} = \frac{1}{u}\cdot\frac{du}{dx}$$

Example 1: $y = \ln(x^2+1)$

$u = x^2+1,\ \frac{du}{dx} = 2x$

$$\frac{dy}{dx} = \frac{1}{x^2+1}\cdot 2x$$

$$\boxed{\frac{dy}{dx} = \frac{2x}{x^2+1}}$$

Example 2: $y = \ln \frac{x}{1+x^2}$

$$y = \ln x - \ln(1+x^2)$$

$u = 1+x^2,\ \frac{du}{dx} = 2x$

$$\frac{dy}{dx} = \frac{1}{x} - \frac{2x}{1+x^2}$$

$$\frac{dy}{dx} = \frac{1-x^2}{x(1+x^2)}$$

These rules also work for trig functions.

Example 3: $y = \ln(\cos(4x))$

$u = \cos(4x),\ \frac{du}{dx} = -\sin(4x)\cdot 4$

$$\frac{dy}{dx} = \frac{1}{\cos(4x)}\cdot -4\sin(4x)$$

$$= \frac{-4\sin(4x)}{\cos(4x)}$$

$$\boxed{\frac{dy}{dx} = -4\tan(4x)}$$

Integration formula:

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

Example 4: $\int \frac{3x^2}{x^3+5}\,dx$ $\quad u = x^3+5,\ du = 3x^2\,dx$

$$= \int \frac{1}{u}\,du$$

$$= \ln|x^3+5| + C$$

Example 5: $\int \frac{x^2}{x^3-4}\,dx$ $\quad u = x^3-4,\ du = 3x^2\,dx$

$$= \frac{1}{3}\int \frac{3x^2\,dx}{x^3-4}$$

$$= \frac{1}{3}\int \frac{1}{u}\,du$$

$$= \frac{1}{3}\ln|x^3-4| + C$$

Assignment: AP Calculus p396 (1-10, 21-30)

F. The Exponential Function e^x - Section 6.6

$$e = 2.718281828459045\ldots$$

The rule for the derivative is:

$$\boxed{\frac{d(e^u)}{dx} = e^u \frac{du}{dx}}$$

Example 1: $y = e^{-2x}$

$u = -2x, \; \frac{du}{dx} = -2$

$$\frac{dy}{dx} = e^{-2x} \cdot -2$$

$$\boxed{\frac{dy}{dx} = -2e^{-2x}}$$

Example 2: $y = e^{x^3}$

$u = x^3; \; \frac{du}{dx} = 3x^2$

$$\frac{dy}{dx} = e^{x^3} \cdot 3x^2$$

$$\boxed{\frac{dy}{dx} = 3x^2 e^{x^3}}$$

Example 3: $y = e^{\cos x}$

$u = \cos x, \; \frac{du}{dx} = -\sin x$

$$\frac{dy}{dx} = e^{\cos x} \cdot -\sin x$$

$$\boxed{\frac{dy}{dx} = -\sin x \cdot e^{\cos x}}$$

The rule for the integral is:

$$\boxed{\int e^x dx = e^x + C}$$

Example 4: $\int e^{5x} dx$

$u = 5x, \; du = 5dx$

$$= \frac{1}{5}\int 5e^{5x} dx$$

$$\boxed{= \frac{1}{5} e^{5x} + C}$$

Example 5: $\int e^x(1+e^x)^{1/2}\,dx$

$u=(1+e^x)$, $du=e^x\,dx$

$= \int u^{1/2}\,du$

$= \dfrac{u^{3/2}}{3/2}+C$

$= \frac{2}{3}u^{3/2}+C$

$\boxed{= \frac{2}{3}(1+e^x)^{3/2}+C}$

Example 6: $\int x^3 e^{3x^4}\,dx$

$u=3x^4$, $du=12x^3\,dx$

$= \frac{1}{12}\int 12x^3 e^{3x^4}\,dx$

$\boxed{= \frac{1}{12}e^{3x^4}+C}$

Assignment: AP Calculus p409 (19-26, 43-48)

6.9 - Application of log. functions.

page 433

1.) $y = y_0 e^{kt}$

y = final pop

y_0 = ini. pop

e = e

k = constant (must find)

t = time (in hours)

at $t=0$, $y=1$

at $t=\frac{1}{2}$, $2 = e^{.5k}$

$\ln 2 = .5k$

$k = 2\ln 2$

$k \doteq 1.3863$

at $t = 24$ hrs.

$y = 1e^{(24)(1.3863)}$

$\boxed{y = 2.8151 \times 10^{14}}$

2.) $y = y_0 e^{kt}$

at $t=3$, $10000 = y_0 e^{3k}$, at $t=5$, $40000 = y_0 e^{5k}$

$40000 = 4y_0 e^{3k}$

$\therefore 4y_0 e^{3k} = y_0 e^{5k}$

$4 = e^{2k}$

$\ln 4 = 2k$

$k = \frac{1.3863}{2} = .6931$

$\therefore 10000 = y_0 e^{2.079}$

$10000 = y_0 (7.996)$

$\boxed{y_0 = 1250.625}$

Ch 7.2

Integration by Parts

Comes from $y = u \cdot v$

$$d(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\Rightarrow u\frac{dv}{dx} = d(u \cdot v) - v\frac{du}{dx}$$

mult by dx

$$\therefore u\,dv = d(u \cdot v)\,dx - v\,du$$

$$\int u\,dv = u \cdot v - \int v\,du$$

Example $\int x\cos(x)\,dx$

$u = x \quad dv = \cos(x)\,dx$

$du = dx \quad v = \sin(x)$

$$\therefore \int x\cos(x)\,dx = x\sin(x) - \int \sin(x)\,dx$$

$$= \boxed{x\sin(x) + \cos(x) + C}$$

Assign: p455 (1-20) odd

E. Trig Substitutions that Replace a^2-u^2, a^2+u^2, and u^2-a^2 by single Squared Terms - Section 7.5

There are 3 substitutions that you must Know (p471):

1.) for a^2-u^2, let $u = a\sin\theta \Rightarrow a^2\cos^2\theta$

2.) for a^2+u^2, let $u = a\tan\theta \Rightarrow a^2\sec^2\theta$

3.) for u^2-a^2, let $u = a\sec\theta \Rightarrow a^2\tan^2\theta$

Example #1: $\int_{-1}^{1} \sqrt{1-x^2}\,dx$

① Let $x = 1\sin\theta$
$dx = \cos\theta\, d\theta$

② find the new limits of integration
$1 = \sin\theta \Rightarrow \theta = \frac{\pi}{2}$
$-1 = \sin\theta \Rightarrow \theta = -\frac{\pi}{2}$
Note: $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

③ $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\,\cos\theta\, d\theta$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos^2\theta}\,\cos\theta\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\,\cos\theta\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tfrac{1}{2}\, d\theta + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tfrac{1}{2}\cos(2\theta)\, d\theta = \tfrac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta + \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos(2\theta)\, d\theta$$

$$= \tfrac{1}{2}\theta + \tfrac{1}{4}\sin(2\theta)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

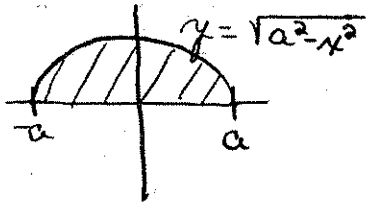
(Note: $\sin 2\theta = 2\sin\theta\cos\theta$ - not needed, but might be interesting to use)

$$= \left[\tfrac{1}{2}\left(\tfrac{\pi}{2}\right) + \tfrac{1}{4}(0)\right] - \left[\tfrac{1}{2}\left(-\tfrac{\pi}{2}\right) + \tfrac{1}{4}(0)\right]$$

$$= \frac{\pi}{4} + \frac{\pi}{4} = \boxed{\frac{\pi}{2}}$$

An interesting note is that this is the area of a semicircle of radius 1, this would give use the answer $\frac{1}{2}(\pi)(1)^2 = \frac{1}{2}\pi$

In general, $\int_{-a}^{a} \sqrt{a^2-x^2}\, dx$



$$\Rightarrow \int_{-a}^{a} \sqrt{a^2-x^2}\, dx = \frac{1}{2}(\pi a^2)$$

This also implies that

$$\int_{0}^{a} \sqrt{a^2-x^2}\, dx = \left(\frac{1}{2}\right)\left[\frac{1}{2}(\pi a^2)\right]$$

For example (#23 on page 476)

$$\int_{0}^{5} \sqrt{25-x^2}\, dx = \left(\frac{1}{2}\right)\left[\frac{1}{2}(\pi(5)^2)\right]$$
$$= \frac{1}{4}(25\pi) = \boxed{\frac{25}{4}\pi}$$

check this by the long way.

Example #2: $\int_{-2}^{2} \frac{dx}{4+x^2}$

① $x = 2\tan\theta$
$dx = 2\sec^2\theta\, d\theta$

② $2 = 2\tan\theta \Rightarrow 1 = \tan\theta \Rightarrow \theta = \frac{\pi}{4}$
$-2 = 2\tan\theta \Rightarrow -1 = \tan\theta \Rightarrow \theta = -\frac{\pi}{4}$

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{2\sec^2\theta\, d\theta}{(4+4\tan^2\theta)} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{2\sec^2\theta\, d\theta}{4(1+\tan^2\theta)} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2\theta\, d\theta}{2\sec^2\theta}$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}\, d\theta = \frac{1}{2}\theta \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{1}{2}\left(\frac{\pi}{4}\right) - \frac{1}{2}\left(-\frac{\pi}{4}\right) = \frac{\pi}{8} + \frac{\pi}{8} = \boxed{\frac{\pi}{4}}$$

Assignment: AP Calculus p 476 (4, 6, 7, 9)

Ch 7.7 - Partial Fractions

The idea is to break the function $\frac{f(x)}{g(x)}$ into smaller fractions to make it easy to integrate.

Example: $\int \frac{1}{x^2+x}\,dx$

we can not do this, therefore we need to change it to something we can work with.

$$\frac{1}{x^2+x} = \frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

mult by the den: $1 = A(x+1) + Bx$

find A+B

$$\text{at } x=0 \Rightarrow 1 = A(1)+0$$

$$A = 1$$

$$\text{at } x=-1 \Rightarrow 1 = 0 + B(-1)$$

$$B = -1$$

$$\therefore \frac{1}{x(x+1)} = \frac{1}{x} + \frac{-1}{x+1}$$

$$= \frac{1}{x} - \frac{1}{x+1}$$

$$\int \frac{1}{x^2+x}\,dx = \int \frac{1}{x}\,dx - \int \frac{1}{x+1}\,dx$$

$$= \ln x - \ln(x+1) + C$$

Assign: p 489 (1-5)